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ABSTRACT

The Sequential Mathematics Sequence provides an integrated course of study for grades 9-11 in New York City. This document presents lesson plans for part one of the second-year course, designed for 10th grade. The curriculum interweaves algebra, geometry, logic, probability, statistics, and trigonometry, with the emphasis in the 61 lessons in this document on logic, trigonometry and probability. Each lesson begins with a clearly stated aim presented in question form. Performance objectives, new vocabulary, and a motivational activity follow. A full development is then presented, with suggested pivotal questions. Applications, a summary, and sample homework examples are also included. In each lesson opportunities are infused for helping students to develop critical thinking, problem-solving, and reasoning skills. (MNS)

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SEQUENTIAL MATHEMATICS

Course II

Part 1



NEW YORK CITY BOARD OF EDUCATION
Office of Curriculum Development and Support
Division of Curriculum and Instruction

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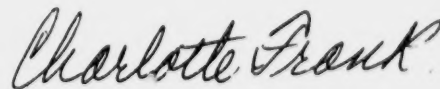
FOREWORD

The Sequential Mathematics Sequence provides New York City students with an integrated course of study for grades nine, ten, and eleven. In a departure from traditional courses of study, lessons interweave algebra, geometry, logic, probability, and mathematical systems. This course extends the integrated approach that begins in kindergarten and continues through grade 8.

In the Sequential Mathematics Sequence, algebra, geometry, and trigonometry remain important components of the curriculum. In addition, other branches of mathematics --such as probability, logic, and mathematical systems -- are introduced and emphasized as important tools for dealing with present and future technological developments.

Each lesson in Course II begins with a clearly stated aim presented in question form. Performance objectives follow; these provide a specific focus for the teacher. The new vocabulary for the lesson is presented. This is followed by a motivational activity. A full development is presented, with suggested pivotal questions. Applications, a summary, and sample homework examples are also included. In each lesson opportunities have been infused for helping students to develop critical thinking, problem-solving, and reasoning skills. These activities will enable students to become more skilled in higher levels of thinking.

The Sequential Mathematics Sequence has been designed to anticipate students' needs for mathematical skills in an increasingly technologically-oriented culture. We anticipate that the Sequential Mathematics Sequence will serve the individual needs of your students and enable them to acquire the mathematics skills they need in the present and for the future.



Charlotte Frank
Executive Director
Division of Curriculum and Instruction

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INTRODUCTION

Sequential Mathematics (Course II), Part I is the second year course of the newly implemented New York State curriculum. This publication from the Division of Curriculum and Instruction is a guide for teachers. It attempts to present the material covered in this course in a way that is most comprehensive and useful to the classroom teacher.

Traditionally, in New York State, mathematics in grades nine, ten, and eleven has consisted of algebra, geometry, and intermediate algebra/trigonometry. Students completing the three-year sequence have been exposed to these topics as relatively independent units. Moreover, students discontinuing their study of mathematics after grade eleven have never been exposed to such timely and practical topics as probability, statistics, and formal logical reasoning. Those terminating their high school mathematics program after grade ten have been given even fewer opportunities for exposure to these topics.

The three-year Sequential Mathematics sequence designed by New York State is a move in the direction of fusing these formerly separate topics and introducing those which have been absent from the course of study.

This curriculum guide is based upon the Scope and Sequence for Sequential Mathematics, Course II, Part I. It consists of lesson plans, which should be considered suggestions rather than prescriptions. Each lesson stresses the development of mathematical concepts, with an emphasis on providing opportunities for the development of problem solving skills, reasoning skills and critical thinking. To help students develop these skills, each lesson contains suggested questions to encourage student investigation and thought. The problems presented in the lessons are designed to be relevant to students' experiences. There is an abundance of application and homework problems. These have been provided to help teachers adapt instruction to the individual differences within the classroom.

Though Course II is written with tenth grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less than or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of material to be stressed, or extra material to be presented.

ACKNOWLEDGMENTS

Sequential Mathematics, Course II, Part I was developed for teachers of Sequential Mathematics as a project of the Mathematics Unit of the division of Curriculum and Instruction, Charlotte Frank, Executive Director.

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Gerald Elgarten, Interim Acting Director of Mathematics, and Maria DeSalvio, Assistant Director of Mathematics, gave impetus and guidance to this project by supervising the writing teams, arranging for the planning and production meetings, and screening the materials that were produced by the writers.

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Lesson 1 - Logic

AIM: What are the rules for connectives?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Write the definitional tables for $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$.
2. Determine if compound sentences are tautologies by using the rules of connectives on truth tables.

VOCABULARY: negation, conjunction, disjunction, implication, conditional, biconditional

MOTIVATION:

Given: p: Mary plays tennis.

q: Johnny goes swimming.

What do the statements $p \wedge q$ and $p \vee q$ mean?

DEVELOPMENT:

1. Review the meaning of $\wedge$ and $\vee$. Elicit that $\wedge$ means "and". Indicate that it is called a conjunction. Ask: When is a conjunction true? Elicit: When both parts are true. Ask: How do we represent a conjunction symbolically? ($p \wedge q$)
2. Ask: What does $p \vee q$ say? Elicit: $p \vee q$ is "p or q" and is called a disjunction. Ask: When is a disjunction true? (When either p or q is true.) Elicit the answer to the motivation question ($p \wedge q$ means "Mary plays tennis and Johnny goes swimming." while $p \vee q$ means "Mary plays tennis or Johnny goes swimming.")
3. Elicit the meaning of the symbol $\sim$, (the negation). Ask: What does $\sim p$ say? (the negation of p) What is true about a statement and its negation? (They have opposite truth values.)
Elicit the truth table for $\sim, \wedge$ and $\vee$.

p	q	$\sim p$	$p \wedge q$	$p \vee q$
T	T			
T	F			
F	T			
F	F			

Elicit that we start with the p and q columns filled in as shown. This gives us all of the possibilities. Now elicit the remainder of the truth table.

p	q	$\sim p$	$p \wedge q$	$p \vee q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

4. Ask: What does $p \rightarrow q$ state? Elicit that this states "if p, then q". Elicit that this is called a conditional statement or sometimes an implication.

Ask: When we combine two conditionals as in

$(p \rightarrow q) \wedge (q \rightarrow p)$, what do we have? (Answer: a biconditional.) What is the short way of writing this?

$(p \leftrightarrow q)$ Continue the truth table to include $p \rightarrow q$ and

$p \leftrightarrow q$. How is a biconditional read? (p if and only if q) These two columns will look like this.

p	q	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

Elicit: A conditional is false only when the first statement is true and the second is false. Also, if the first statement is false, the conditional is always true.

Ask: When is a biconditional true? (When both statements are true or both are false.)

Note: For classes which do not recall this, it may be necessary to use $(p \rightarrow q) \wedge (q \rightarrow p)$ to achieve the result for the biconditional.

5. Consider the statement $(p \wedge q) \rightarrow p$. How would we determine its truth value? Elicit that a truth table is needed. Again elicit the table as follows:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Ask: What do we notice about its truth value? (always true) What is such a statement called? (tautology)
Elicit and write the definition: A tautology is a statement that is always true.

Ask: Is $p \vee (\sim p)$ a tautology? Elicit that to determine whether any statement is a tautology, we construct a truth table. Have the students construct this one by themselves.

Solution:

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

APPLICATIONS:

- Write each statement symbolically. Use:

p: John is tall.

q: John has red hair.

r: Mary has blond hair.

s: Yvonne is a cheerleader.

- John is tall and Yvonne is a cheerleader.
- If John has red hair then he is tall.
- Yvonne is a cheerleader if and only if Mary has blond hair.
- John is not tall.
- John has red hair or Mary has blond hair.
- If John is tall then he doesn't have red hair.

Answers:

a) $p \wedge s$

d) $\sim p$

b) $q \rightarrow p$

e) $q \vee r$

c) $s \leftrightarrow r$

f) $p \rightarrow (\sim q)$

- Write a truth table and determine whether each of the following is a tautology.

a) $(p \vee q) \rightarrow p$ Answers: no

b) $(r \wedge s) \rightarrow s$ yes

c) $(p \vee q) \rightarrow (p \wedge q)$ no

d) $[(p \rightarrow q) \wedge \sim q] \rightarrow p$ no

SUMMARY:

1. What do the symbols $\sim$, $\wedge$, $\vee$, $\longrightarrow$ and $\longleftrightarrow$ mean?
2. How do we determine if a compound statement is a tautology?

HOMEWORK:

1. Using the following, translate each sentence into words:

p: $4 + 5 > 7$

q: A square is a rectangle.

r: Michael Jackson wears one glove.

s: Brown cows give chocolate milk.

Answers:

- | | |
|-------------------------------|---|
| a) $p \longrightarrow q$ | a) If $4 + 5 > 7$, then a square is a rectangle. |
| b) $q \vee r$ | b) A square is a rectangle or Michael Jackson wears one glove. |
| c) $r \longleftrightarrow s$ | c) Michael Jackson wears one glove if and only if brown cows give chocolate milk. |
| d) $r \wedge \sim p$ | d) Michael Jackson wears one glove and $4+5 \leq 7$. |
| e) $\sim s \longrightarrow q$ | e) If brown cows don't give chocolate milk then a square is a rectangle. |

2. Using the sentences in 1 above, express each of the following symbolically and determine its truth value.
 - a) $4 + 5 > 7$ or Michael Jackson wears one glove.
 - b) If brown cows give chocolate milk then a square is a rectangle.
 - c) A square is not a rectangle and $4 + 5 > 7$.
 - d) Michael Jackson wears one glove if and only if brown cows don't give chocolate milk.

Answers:

- | | | | |
|--------------------------|------|-------------------------------------|---|
| a) $p \vee r$ | True | c) $\sim q \wedge p$ | False |
| b) $s \longrightarrow q$ | True | d) $r \longleftrightarrow (\sim s)$ | True when M.J. wears one glove.
False otherwise. |

3. Determine whether each of the following is a tautology.

Answers:

- | | |
|---|--------|
| a) $\sim q \longrightarrow q$ | a) no |
| b) $p \longrightarrow (p \vee q)$ | b) yes |
| c) $\sim (p \vee q) \longleftrightarrow (\sim p \wedge \sim q)$ | c) yes |

Lesson 2 - Logic

AIM: What are some of the laws of logic?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State: $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$ have the same truth value.
(Law of Contrapositive)

2. State: If p is true and if p implies q , then q is true.

Symbolically:
$$\begin{array}{c} p \rightarrow q \\ p \\ \hline q \end{array}$$

(Law of Detachment or Modus Ponens)

3. State: If p implies q and q is false, then p is false.

Symbolically:
$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \sim p \end{array}$$

(Modus Tollens or contrapositive inference)

4. Draw conclusions using the above laws of logic.

VOCABULARY: detachment, modus ponens, modus tollens,
contrapositive inference

MOTIVATION:

1. Write the contrapositive of

a) $p \rightarrow q$

b) $\sim q \rightarrow \sim p$

2. Which of the following statements are logically equivalent?

a) If it rains, I go to the movies.

b) If I go to the movies, it rains.

c) If I don't go to the movies, then it doesn't rain.

DEVELOPMENT:

1. Elicit the contrapositive of the statements in 1 above.

The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$. This shows that the contrapositive of $\sim q \rightarrow \sim p$ is $p \rightarrow q$. Elicit that these are logically equivalent.

This can be stated as a law of logic: The law of

contrapositive states $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$ have the same truth value. Elicit that therefore these are now interchangeable.

Elicit the answer to question 2 above: a) and c) are contrapositives of each other and are logically equivalent.

2. Present the conditional statement: If Mary does her homework then she will pass. State: Mary does her homework. Ask: What conclusion can we draw? (Mary will pass.) Using p and q , how can we state this symbolically? Answer: If both $(p \rightarrow q)$ and p are true, then q is true. That is, when we are given if p , then q , and we know p is true, then q is true. This is called the law of detachment or modus ponens.

3. Suppose both $p \rightarrow q$ and $\sim q$ are true. Can we draw a conclusion? From the law of contrapositive, $(p \rightarrow q)$ is logically equivalent to $(\sim q \rightarrow \sim p)$. From the law of detachment, if both $(\sim q \rightarrow \sim p)$ and $\sim q$ are true, then we conclude $\sim p$ is true. Elicit that this will always be true. Hence we have a third law. This law is called modus tollens or the law of contrapositive inference. Ask: How would we write this law symbolically? Elicit:

$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \sim p \end{array}$$

4. Give the students the following statement: If my mother drives me to school, then I am ill. Ask: What conclusion can we draw? (That if I am not ill, my mother won't drive me to school.) What law tells us this is a valid conclusion? (law of contrapositive) Ask: How would we write this symbolically? $\frac{p \rightarrow q}{\sim q \rightarrow \sim p}$

Ask the class to consider the following premises:

$$\left. \begin{array}{l} P_1: r \rightarrow s \\ P_2: r \end{array} \right\} \begin{array}{l} \text{What conclusion can we draw? } (s) \\ \text{Why? (law of detachment)} \end{array}$$

Now consider the statements: If I drive after drinking,

I will have an accident; and I did not have an accident.
 What conclusion can we draw? (I did not drive after
 drinking.) What law would justify this answer?
 (contrapositive inference) How would we write this
 symbolically?
$$\frac{p \rightarrow q \quad \sim q}{\sim p}$$

5. Elicit a medial summary: What laws of logic can we use to draw conclusions?

APPLICATIONS:

1. Draw a conclusion from the given premises.

Answers:

- | | |
|--------------------------------|--------------------------------|
| a) $P_1: p \rightarrow q$ | a) q |
| $P_2: p$ | |
| b) $P_1: p \rightarrow \sim r$ | b) $\sim r$ |
| $P_2: p$ | |
| c) $P_1: s \rightarrow t$ | c) $\sim t \rightarrow \sim s$ |
| d) $P_1: q \rightarrow r$ | d) $\sim q$ |
| $P_2: \sim r$ | |
| e) $P_1: \sim s \rightarrow t$ | e) s |
| $P_2: \sim t$ | |

2. Determine if the following conclusions are valid and, if so, state the law of inference that applies.

Answers:

- | | |
|------------------------------------|-----------------------------|
| a) $P_1: r \rightarrow s$ | a) valid/law of |
| concl: $\sim s \rightarrow \sim r$ | contrapositive |
| b) $P_1: q \rightarrow t$ | b) not valid |
| $P_2: t$ | (reasoning from converse) |
| concl: q | |
| c) $P_1: q \rightarrow \sim r$ | c) law of detachment (modus |
| $P_2: q$ | ponens) |
| concl: $\sim r$ | |

- d) P_1 : If I like music, I like Michael Jackson.
 P_2 : I don't like Michael Jackson.
 concl: I don't like music.
 Answer: valid/contrapositive inference (modus tollens)
- e) P_1 : If $3 + 5 > 7$ then $2 + 6 < 9$
 P_2 : $2 + 6 < 9$
 concl: $3 + 5 > 7$
 Answer: not valid (reasoning from converse)

3. Draw a valid conclusion and state the law supporting that conclusion.

- a) P_1 : If birds fly then fish swim.
 P_2 : Birds fly
 Answer: Fish swim - law of detachment
- b) P_1 : If goobers giggle the wibbles wobble.
 Answer: If wibbles don't wobble then goobers don't giggle - law of contrapositive
- c) P_1 : If Tom lives in N.Y.C., then he is fresh.
 P_2 : Tom is not fresh.
 Answer: Tom doesn't live in N.Y.C. - contrapositive inference
- d) P_1 : Mary failed her exam.
 P_2 : If Mary studies, she will pass her exam.
 Answer: Mary did not study - contrapositive inference

SUMMARY:

1. What do we use to draw conclusions in logic? (laws of logic)
2. What does the law of detachment allow us to conclude?

HOMEWORK:

1. Draw a conclusion and state the law that you used to justify it.

- a) $P_1: s \rightarrow t$ $P_2: s$
Answer: t - law of detachment
- b) $P_1: q \rightarrow p$ $P_2: \sim p$
Answer: $\sim q$ - law of contrapositive inference
- c) P_1 : If Jane does poorly in school she will be grounded.
 P_2 : Jane was not grounded.
Answer: Jane didn't do poorly in school - contrapositive inference.
- d) P_1 : Football players are good students.
Answer: If he is not a good student, then he is not a football player - law of contrapositive.

2. Determine whether the conclusion is valid or not. If it is valid, state the law used.

- a) $P_1: p \rightarrow r$ $P_2: r$ $\text{concl: } p$
Answer: not valid
(reasoning from converse)
- b) $P_1: s \rightarrow \sim q$ $P_2: s$ $\text{concl: } \sim q$
Answer: law of detachment
- c) P_1 : If Jane is a maniac then she is a freak.
 P_2 : Jane is not a freak.
 $\text{concl: } \text{Jane is not a maniac.}$
Answer: valid - contrapositive inference
- d) P_1 : If 7 is prime then 11 is lucky.
 P_2 : 11 is lucky
 $\text{concl: } 7 \text{ is not prime.}$
Answer: invalid reasoning

Lesson 3 - Logic

AIM: How can we apply laws of logic to test the validity of an argument?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Write simple proofs (2 columns with statements and reasons or discursive) using the Laws of Contrapositive, Detachment and Modus Tollens (Contrapositive Inference).
2. Use above proofs to test the validity of conclusions.

VOCABULARY: proof, statement, reason, valid, conclusion

MOTIVATION:

Do you agree with the conclusion, given the following statements?

- 1) If the singer is Michael Jackson, then he is a great entertainer.
- 2) The singer is Michael Jackson.

Conclusion: He is a great entertainer.

DEVELOPMENT:

1. The conclusion is valid. How can we show this?

Translate the sentences symbolically.

$$P_1 = p \rightarrow q$$

$$P_2 = p$$

Conclusion: q What law is this? (law of detachment)

2. Consider the following premises:

P_1 : If Tom eats pizza, he will be thirsty.

P_2 : Tom is not thirsty.

What conclusion can be drawn? Tom didn't eat pizza.

Let us write this symbolically: $P_1: p \rightarrow q$

$$P_2: \sim q$$

Conclusion: $\sim p$ (by Modus

Tollens)

Now let's consider these premises and a conclusion.

P_1 : All sophomores love Culture Club.

P_2 : Jennifer is not a sophomore.

Conclusion: Jennifer does not love Culture Club.

Ask: Is this a valid argument? (no) Show that this does not follow any of the laws of inference.

Write the premises symbolically - P_1 : $p \rightarrow q$

P_2 : $\sim p$

Conclusion: $\sim q$

Note: This is not the law of contrapositive inference. It is reasoning from inverse which is involved, and not valid.

Ask: What would have had to be changed to use this law? (Premise 2 would have to be $\sim q$ and the conclusion would have to be $\sim p$.)

3. Let us consider P_1 : $r \rightarrow s$

P_2 : $\sim s$

Conclusion: $\sim r$

We would like to write a formal explanation. We call it a proof. We usually write formal proofs in two columns.

That is:

Statements	Reasons
1. $r \rightarrow s$	1. given
2. $\sim s \rightarrow \sim r$	2. law of contrapositive
3. $\sim s$	3. given
4. $\sim r$	4. law of detachment

Explain to the students that the first statement will be a given premise, and the first reason will be "given". That is, the reason we know the statement is true is because it was given. Therefore, the proof can be as follows.

Statements	Reasons
1. $r \rightarrow s$	1. given
2. $\sim s \rightarrow \sim r$	2. law of contrapositive
3. $\sim s$	3. given
4. $\sim r$	4. law of detachment

Elicit that there is another way to do this using only three statements and reasons. Elicit the steps:

Statements	Reasons
1. $r \rightarrow s$	1. given
2. $\sim s$	2. given
3. $\sim r$	3. contrapositive inference (or Modus Tollens)

Now try another one. First draw the conclusion, then write a proof.

$$P_1: \sim q \rightarrow w$$

$$P_2: \sim q$$

Elicit the conclusion w , then elicit the proof.

Statements	Reasons
1. $\sim q \rightarrow w$	1. given
2. $\sim q$	2. given
3. w	3. law of detachment

4. Now let's try this with statements. Consider: If he is tall, he will make the basketball team; and Jack did not make the basketball team.

Elicit the conclusion: Jack is not tall. Ask: What would we have to do to prove this? Elicit: First write the premises and conclusion symbolically. Let:

t : He is tall.

b : He will make the basketball team.

Elicit: $P_1: t \rightarrow b$

$$P_2: \sim b$$

Now elicit the conclusion, $\sim t$, and the proof.

Statements	Reasons
1. $t \rightarrow b$	1. given
2. $\sim b$	2. given
3. $\sim t$	3. modus tollens (contrapositive inference)

APPLICATIONS:

1. Determine whether or not the conclusion is valid. If so, state the law of inference. If not, write invalid.

a) $P_1: p \rightarrow r$

$$P_2: p$$

concl: r

ans: valid

law of detachment

b) $P_1: s \rightarrow t$

$$\text{concl: } \sim t \rightarrow \sim s$$

ans: valid

law of contrapositive

- c) $P_1: q \rightarrow r$
 $P_2: r$
concl: q
ans: invalid
(reasoning from converse)
- d) $P_1: x \rightarrow y$
 $P_2: \sim y$
concl: $\sim x$
ans: valid/law of contrapositive inference
- e) $P_1: \text{If } 3 + 5 > 7, \text{ then the sun will shine.}$
 $P_2: \text{The sun won't shine.}$
concl: $3 + 5 \not> 7$
ans: valid/law of contrapositive inference

2. Prove each of the following with a two column proof.

- a) $P_1: \text{If today is Monday, then tomorrow is Tuesday.}$
 $P_2: \text{Tomorrow is not Tuesday.}$
concl: Today is not Monday.

Answer: Let p : Today is Monday. $P_1: p \rightarrow q$
 q : Tomorrow is Tuesday. $P_2: \sim q$
Conclusion: $\sim p$

Statements	Reasons
1. $p \rightarrow q$	1. given
2. $\sim q$	2. given
3. $\sim p$	3. law of contrapositive inference

- b) $P_1: \text{If it waddles it is a duck.}$
 $P_2: \text{Horace waddles.}$
concl: Horace is a duck.

Answer: $P_1: w \rightarrow d$
 $P_2: w$
concl: d

Statements	Reasons
1. $w \rightarrow d$	1. given
2. w	2. given
3. d	3. law of detachment

3. $P_1: m \rightarrow t$
 $P_2: \sim t$
concl: $\sim m$
4. $P_1: \sim t \rightarrow u$
 $P_2: \sim t$
concl: u
- 3 and 4 are done similarly.

SUMMARY:

1. How do we write a formal proof?
2. With what statement will a proof begin?
3. With what statement will a proof conclude?

HOMEWORK:

Determine whether or not each conclusion is valid. If it is, write a proof for it. If it is not valid, write invalid.

- | | |
|---|--|
| 1. $P_1: x \rightarrow y$
$P_2: x$
concl: y | 2. $P_1: \sim q \rightarrow p$
$P_2: \sim p$
concl: $\sim q$ |
| 3. $P_1: \sim s \rightarrow \sim t$
$P_2: t$
concl: s | |
| 4. P_1 : If it is Friday, then I am happy.
P_2 : I am happy.
concl: It is Friday. | |
| 5. P_1 : The capitol of Nebraska is Lincoln.
P_2 : The capitol of this state is not Lincoln.
concl: This state is not Nebraska. | |

Answers:

1. valid

Statements	Reasons
1. $x \rightarrow y$	1. given
2. x	2. given
3. y	3. law of detachment

2. invalid

3. valid - law of contrapositive inference

4. invalid - reasoning from converse

5. valid - law of contrapositive inference

Lesson 4 - Logic

AIM: What are the Laws of Syllogism and Disjunctive Inference?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Use simple sentences to establish intuitively

a) The Law of Syllogism (chain rule):

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

b) Laws of Disjunctive Inference:

$$\frac{p \vee q}{\sim p} \text{ or } \frac{p \vee q}{\sim q} \quad \frac{\quad}{q} \quad \frac{\quad}{p}$$

2. Draw simple conclusions using the above laws.

VOCABULARY: syllogism, chain rule, disjunctive inference

MOTIVATION:

Draw a valid conclusion from

1. If Sylvia does her work, then Joan can finish her job.
2. If Joan finishes her job, then Herb will be happy.

DEVELOPMENT:

1. Elicit the conclusion: If Sylvia does her work, then Herb will be happy.

Ask the class to consider the following:

If it is October, then the leaves turn colors. And if the leaves turn colors, then they will soon fall from the trees. What conclusion can be drawn? (If it is October, then the leaves will soon fall from the trees.) Explain that this is the law of syllogism or the chain rule.

Elicit the chain rule symbolically:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Ask the class to draw a conclusion from $p \rightarrow q$ and $q \rightarrow s$. Conclusion: $p \rightarrow s$.

2. Have the class consider the following:

Either it is September or October and it is not October. Ask: What conclusion could you draw? It is September.

Also consider:

Billy will eat either pizza or hot dogs and Billy did not eat pizza. What conclusion is valid? Billy ate hot dogs. Suppose we were told that Billy didn't eat hot dogs, would there be another valid conclusion? (Yes, Billy ate pizza.) Elicit these symbolically.

$$\begin{array}{cc} p \vee q & p \vee q \\ \sim q & \sim p \\ \hline p & q \end{array}$$

Try to elicit that this law deals with the disjunction and is called the disjunctive inference.

3. Now let's try to draw conclusions using these laws.

Consider: If I do my homework, then I will do well on my tests, and if I do well on my tests, then I will receive a good grade. What conclusion should we draw? (If I do my homework, then I will receive a good grade.) Elicit the symbolic representation:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Have a student read this aloud. (If p implies q and q implies r , then p implies r .)

Ask: What law does this represent? (law of syllogism or chain rule)

Present the following premises. Mr. C will teach health or he will coach football. Mr. C does not teach health. What conclusion can be drawn? (Mr. C. coaches football.)

Elicit the symbolic representation: $(p \vee q)$ and $\sim p$, therefore q . Ask: What law of inference is this? (law of disjunctive inference)

APPLICATIONS:

1. Draw a conclusion from the following:

- a) $P_1: p \rightarrow q$ b) $P_1: s \vee t$
 $P_2: q \rightarrow r$ $P_2: \sim s$
Ans: $p \rightarrow r$ Ans: t
- c) $P_1: \text{He will eat lasagna or ravioli.}$
 $P_2: \text{He didn't eat ravioli.}$
Ans: He ate lasagna.
- d) $P_1: \text{If donkeys fly then horses sing.}$
 $P_2: \text{If horses sing then birds swim.}$
Ans: If donkeys fly then birds swim.

2. Determine if the conclusion is valid and if so, state the reason; if not, state INVALID.

- a) $P_1: x \rightarrow y$ b) $P_1: s \vee t$
 $P_2: y \rightarrow z$ $P_2: s$
concl: $x \rightarrow z$ concl: t
ans: VALID (law of ans: INVALID
syllogism or chain rule)
- c) $P_1: \text{Sandy sells sea shells or she doesn't go dancing.}$
 $P_2: \text{Sandy goes dancing.}$
concl: Sandy sells sea shells.
ans: VALID (law of disjunctive inference)
- d) $P_1: \text{If I go north, I will get to Peekskill.}$
 $P_2: \text{If I get to Peekskill, the trees will be bare.}$
concl: If I go north, the trees will be bare.
ans: VALID (law of syllogism or chain rule)

SUMMARY:

1. What is the symbolic representation of the law of syllogism?

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

2. What is an example of the law of disjunctive inference?

HOMEWORK:

1. Draw a conclusion and state the law of inference that justifies it.

- a) $P_1: p \vee q$ b) $P_1: q \rightarrow r$
 $P_2: \sim q$ $P_2: r \rightarrow \sim p$
Ans: p : law of Ans: $q \rightarrow \sim p$: chain rule
disjunctive inference
- c) P_1 : If it is hot then I go to the beach.
 P_2 : If I go to the beach, I play volleyball.
Ans: If it is hot, then I play volleyball.
- chain rule
- d) P_1 : If it is raining, I will wear a raincoat.
 P_2 : I am not wearing a raincoat.
Ans: It is not raining. - contrapositive inference
(This exercise reviews concepts from lesson 3.)
- e) $P_1: p \rightarrow q$
 $P_2: q \rightarrow s$
 $P_3: p$
Ans: s Note: This uses both detachment and chain rule.

2. Draw a conclusion:

- a) If $ab = 0$ then $a \vee b = 0$. $a \neq 0$.
b) If $ab = 1$ and $a = \frac{2}{3}$, then $b = \underline{\hspace{1cm}}$.
c) If $ab < 0$ and $a > 0$, then $b = \underline{\hspace{1cm}}$.

Lesson 5 - Logic

AIM: How can we use the laws of logic to test the validity of an argument?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the laws of logic to advertisements and political statements to test if an implied conclusion is valid.
2. Write simple proofs to validate such statements.

VOCABULARY: advertisement, disjunctive inference

MOTIVATION:

An advertisement claims that: "If you use Trial soap, your skin will be beautiful." They then show a picture of someone who doesn't use Trial. What conclusion do they want you to draw?

Is this valid?

DEVELOPMENT:

1. Consider the motivation question. Have students write the statements symbolically.

1) $p \rightarrow q$

2) $\sim p$

Is it logical to conclude $\sim q$?

Suppose they showed a picture of someone who does use Trial soap. What conclusion would they want you to draw?

Answer: She has beautiful skin. Where is the problem?

Discuss that while the argument that if both

$(p \rightarrow q)$ and p are true then q is true, the conclusion does not follow. Discuss other ramifications of the advertisement as well.

2. Consider $p \rightarrow q$ and $q \rightarrow r$. Does $p \rightarrow r$? (Yes; chain rule or law of syllogism)

Let us try to write a proof involving the following:

- | | |
|----------------------|---|
| 1) $p \rightarrow q$ | Given these four premises,
can we prove s? |
| 2) $q \rightarrow s$ | |
| 3) $p \vee r$ | |
| 4) $\sim r$ | |

Let us form two columns as before. Elicit the headings:

Statements	Reasons
What should we use first? (a given premise)	
1. $p \rightarrow q$	1. given
2. $q \rightarrow s$	2. given
What can we now state? Why?	
3. $p \rightarrow s$	3. chain rule
What must we now use?	
4. $p \vee r$	4. given
5. $\sim r$	5. given
What conclusion can we draw from 4 and 5?	
6. p	6. law of disjunctive inference
What can we now conclude (from 3 and 6)?	
7. s	7. law of detachment

3. Let's consider another problem. We know:

1. Either Mary is slim or tall.
2. If Mary is slim, then she will be a model.
3. Mary is not tall.
4. If Mary is a model then she will be on T.V.

Therefore, Mary will be on T.V.

What would be the first step in proving that the conclusion is true? First write each sentence symbolically. Let:

- | | |
|---------------------|-------------------------|
| S: Mary is slim. | T: Mary is tall. |
| M: Mary is a model. | V: Mary will be on T.V. |

Using these, elicit the symbolic representation of each of the premises.

- | | |
|---------------|----------------------|
| 1. $S \vee T$ | 2. $S \rightarrow M$ |
| 3. $\sim T$ | 4. $M \rightarrow V$ |

Conclusion: V.

What would be the next step? Write two columns.

Statements	Reasons
How should we begin?	
1. $S \vee T$	1. given
2. $\sim T$	2. given
When we see this, what law should come to mind? (disjunctive inference)	
3. S	3. law of disjunctive inference
What would come next?	
4. $S \rightarrow M$	4. given
5. $M \rightarrow V$	5. given
From 4 and 5, what can we conclude?	
6. $S \rightarrow V$	6. law of syllogism
Can we now conclude V?	
7. V	7. law of detachment

Note: For slower classes use steps 1-3 and/or 4-6 as an entire proof. These will only include one law each.

APPLICATIONS:

- Write a proof for each of the following.

- a) $P_1: r \rightarrow s$
 $P_2: r \vee t$
 $P_3: \sim t$
concl: s

Statements	Reasons
1. $r \vee t$	1. given
2. $\sim t$	2. given
3. r	3. disjunctive inference
4. $r \rightarrow s$	4. given
5. s	5. law of detachment

- b) $P_1: q \rightarrow p$
 $P_2: \sim q \rightarrow r$
concl: $\sim p \rightarrow r$

Statements	Reasons
1. $q \rightarrow p$	1. given
2. $\sim p \rightarrow \sim q$	2. contrapositive
3. $\sim q \rightarrow r$	3. given
4. $\sim p \rightarrow r$	4. chain rule (or law of syllogism)

c) P_1 : If Mary has a little lamb, its fleece is white as snow.

P_2 : Mary has a little lamb.

concl: Its fleece is white as snow.

ans: law of detachment

d) P_1 : If you use Vial soap, you will smell like a rose.

P_2 : Jane doesn't smell like a rose.

concl: Jane doesn't use Vial soap.

ans: contrapositive inference

SUMMARY:

1. How should a proof begin?
2. With what statement will it conclude?

HOMEWORK:

1. Determine if each of the following arguments is valid. If so, state the reason.

a) P_1 : If you brush your teeth with BJAX, you will have no teeth.

P_2 : You have teeth.

concl: You don't brush with BJAX.

ans: valid, contrapositive inference

b) P_1 : If Mr. Regal is elected, he won't raise taxes.

P_2 : If I vote for Mr. Regal, he will get elected.

concl: If I vote for Mr. Regal, he won't raise taxes.

ans: valid, chain rule

c) $P_1: p \vee q$
 $P_2: p$
concl: q
ans: invalid

2. Write a formal proof for each of the following.

a) $P_1: p \rightarrow q$
 $P_2: q \rightarrow r$
 $P_3: \sim r$
concl: $\sim p$

b) $P_1: s \vee t$
 $P_2: s \rightarrow r$
 $P_3: \sim t$
concl: r

Answer:

2a)	Statements	Reasons
	1. $p \rightarrow q$	1. Given
	2. $q \rightarrow r$	2. Given
	3. $p \rightarrow r$	3. chain rule
	4. $\sim r$	4. given
	5. $\sim p$	5. contrapositive inference

Answer:

2b)	Statements	Reasons
	1. $s \vee t$	1. given
	2. $\sim t$	2. given
	3. s	3. disjunctive inference
	4. $s \rightarrow r$	4. given
	5. r	5. law of detachment

Lesson 6 - Logic

AIM: How do we negate conjunctions and disjunctions?
(DeMorgan's Laws for negation)

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the laws for the negation of conjunctions and disjunctions (DeMorgan's Laws).
 - a) $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$ have the same truth value.
 - b) $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ have the same truth value.
2. State these laws using simple English sentences.
3. Form the negation of conjunctions and disjunctions.

VOCABULARY: negation, conjunction, disjunction, DeMorgan's Laws

MOTIVATION:

1. How do we determine if one statement is the negation of another? (If they always have opposite truth values - when one is true the other is false and vice versa.)
2. Determine the truth value of the following statement:
Keith Hernandez is a professional baseball player or he plays for the New York Yankees.

DEVELOPMENT:

1. Elicit that this is a true statement and that it is a disjunction. (Note: It is true since in a disjunction only one part must be true and Keith Hernandez is a professional baseball player. Also note he doesn't play for the New York Yankees.) Ask the class to negate this statement. Elicit that the negation must be a false statement since this is a true statement. The negation is: Keith Hernandez is not a professional baseball player AND he doesn't play for the New York Yankees. (Note: The students may present a similar statement using OR instead of AND. Show that now the second part is true and therefore the entire statement is true, which

would be incorrect.) Elicit the symbolic representation. We began with $\sim(p \vee q)$. The negation derived is $(\sim p) \wedge (\sim q)$. Ask: How do we negate a disjunction? (Negate both parts and form a conjunction.) In general, $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ have the same truth value. Elicit this law in simple English. The negation of p or q is the negation of p and the negation of q. Let's try: negate $q \vee (\sim r)$. Elicit that the negation is $(\sim q) \wedge r$.

2. Suppose we began with a conjunction. Cows give milk and roosters lay eggs. Elicit the truth value of this statement. (It is false.) Ask: Why is it a false statement? Elicit that it is a conjunction and a conjunction is only true when both parts are true. (Note: Roosters don't lay eggs. They are males.) Ask: What would be the negation of this statement? Elicit that since this is a false statement, the negation must be true. The proper negation is cows don't give milk or roosters don't lay eggs. This is a true statement since roosters don't lay eggs is true (a disjunction is true if one part is true).
Note: The students may offer cows give milk or roosters lay eggs. This is a true statement. If this occurs, present cows give milk and roosters lay eggs. Here they will see that the negation must follow the pattern of the first one presented. Elicit the symbolic representation to negate $(p \wedge q)$ (we get $(\sim p) \vee (\sim q)$) and a statement to express this in words. (To negate a conjunction, negate both parts and form a disjunction.)
3. Indicate that these two conclusions are stated as laws which are called DeMorgan's Laws and write: $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$ have the same truth value, $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ have the same truth value.
4. Let us form negations using these rules.
 - a) Negate $(\sim p) \wedge q$. By the rule we negate both parts and form a disjunction. Therefore the negation is $p \vee (\sim q)$.
 - b) Negate: Birds swim or fish fly. Again we must

negate both parts and change to a conjunction.
That is: Birds don't swim and fish don't fly.

5. Elicit a medial summary and write it on the board. To negate a disjunction or a conjunction, we negate both parts and change from an and statement to an or statement or vice versa.

APPLICATIONS:

1. Form the negation of each of the following.

- | | |
|---|--------------------------|
| a) $p \vee q$ | c) $(\sim r) \vee q$ |
| Ans: $(\sim p) \wedge (\sim q)$ | Ans: $r \wedge (\sim q)$ |
| b) $s \wedge t$ | d) $x \wedge (\sim y)$ |
| Ans: $(\sim s) \vee (\sim t)$ | Ans: $(\sim x) \vee y$ |
| e) Lawrence Taylor plays for the Jets or the Giants. | |
| Ans: L.T. doesn't play for the Jets and he doesn't play for the Giants. | |
| f) $(6 + 9 = 14) \vee (3 = 2 + 1)$ | |
| Ans: $(6 + 9 \neq 14) \wedge (3 \neq 2 + 1)$ | |
| g) Charlie loves Debbie and John loves Carol. | |
| Ans: Charlie doesn't love Debbie or John doesn't love Carol. | |
| h) Ducks waddle or fish can't swim. | |
| Ans: Ducks don't waddle and fish can swim. | |
| i) The sky is blue and Monday doesn't follow Tuesday. | |
| Ans: The sky is not blue or Monday follows Tuesday. | |

SUMMARY:

1. What must be done to negate a disjunction?
2. How do we negate a conjunction?

HOMEWORK:

Form the negation of each of the following.

- | | |
|-------------------------------|---------------------------------|
| 1. $x \vee z$ | Ans: $(\sim x) \wedge (\sim z)$ |
| 2. $(\sim s) \vee t$ | Ans: $s \wedge (\sim t)$ |
| 3. $(\sim p) \wedge (\sim q)$ | Ans: $p \vee q$ |

4. He loves football and basketball.

Ans: He doesn't love football or he doesn't love basketball.

5. She wears Sergio's or Serrucci's.

Ans: She doesn't wear Sergio's and she doesn't wear Serrucci's.

6. He doesn't like apples but he loves apple pie.

Ans: He likes apples or he doesn't love apple pie.

(Note: "but" is used here in place of "and".)

Lesson 7 - Logic

AIM: How do we use the laws of logic to write more complicated proofs?

PERFORMANCE OBJECTIVES: The student will be able to...

Recall and apply laws of logic to write more complicated proofs by

1. Translating sentences into variables with connectives.
2. Reasoning intuitively with symbols to test the validity of conclusions.
3. Writing proofs similar to those in lesson 3 but requiring more steps.

VOCABULARY: translate, connectives, logically equivalent, plan

MOTIVATION:

Given the following sentences, is the conclusion true?

If Mary buys a stereo, she will build a cabinet.

Mary will buy a V.C.R. or a stereo.

If Mary builds a cabinet, she will quit her job.

Mary does not quit her job.

If Mary buys a V.C.R. she will need tapes.

Conclusion: Mary will need tapes.

DEVELOPMENT:

1. Elicit that in order to know whether this is a true (valid) conclusion, we will try to prove it.

Ask: What must we do first? Elicit: Write each sentence symbolically. Let us use:

s: Mary buys a stereo.

c: She will build a cabinet.

v: Mary will buy a V.C.R.

q: Mary quits her job.

t: Mary will need tapes.

Therefore we get:

$P_1: s \rightarrow c$

$P_2: v \vee s$

$P_3: c \rightarrow q$

$P_4: \sim q$

$P_5: v \rightarrow t$

What is the conclusion? t

In order to get t what would we need to get? Elicit: If we can get v , then using the law of detachment and premise 5, $v \rightarrow t$, we would have t . Therefore we start with a plan. Ask what will be the last statement of our proof? Elicit the conclusion that is t . Therefore our plan begins with t . Plan: t

v
 $v \rightarrow t$
 $\sim s$
 $v \vee s$

That is, if we can get $\sim s$, we can use the law of disjunctive inference. Let us begin our proof. What is the format? Elicit: We list statements and reasons in two columns.

Statements	Reasons
Elicit that we begin with given statements.	
1. $c \rightarrow q$	1. given
2. $\sim q$	2. given
What can we now state? Why?	
3. $\sim c$	3. law of contrapositive inference (modus tollens)
Elicit that from this and premise 1 we get	
4. $s \rightarrow c$	4. given
5. $\sim s$	5. modus tollens
From here we use our plan to finish.	
6. $v \vee s$	6. given
7. v	7. disjunctive inference
8. $v \rightarrow t$	8. given
9. t	9. law of detachment (modus ponens)

Let's review the process for doing logic proofs.

- 1) Translate sentences into symbolic form.
- 2) Write a plan beginning with the conclusion.

Answer:

a)	Statements	Reasons
1.	$a \rightarrow \sim b$	1. given
2.	a	2. given
3.	$\sim b$	3. law of detachment
4.	$c \rightarrow b$	4. given
5.	$\sim c$	5. contrapositive inference
6.	$d \vee c$	6. given
7.	d	7. disjunctive inference

Answer:

b)	Statements	Reasons
1.	$r \rightarrow (\sim q)$	1. given
2.	$q \rightarrow (\sim r)$	2. law of contrapositive
3.	$p \rightarrow q$	3. given
4.	$p \rightarrow (\sim r)$	4. chain rule
5.	p	5. given
6.	$(\sim r)$	6. law of detachment
7.	$r \vee s$	7. given
8.	s	8. disjunctive inference

c) Four horses are in a race. The following statements are true:

1. Ace is first or Trigger is second.
2. If Trigger is second then Silver is third.
3. If Silver is third then Paint is fourth.
4. Paint is not fourth.

concl: Ace is first.

Let A: Ace is first.

T: Trigger is second.

S: Silver is third.

P: Paint is fourth.

First write symbolically.

$P_1: A \vee T$

$P_2: T \rightarrow S$

$P_3: S \rightarrow P$

$P_4: \sim P$

concl: A

Answer:

Statements	Reasons
1. $T \rightarrow S$	1. given
2. $S \rightarrow P$	2. given
3. $T \rightarrow P$	3. chain rule
4. $\sim P$	4. given
5. $\sim T$	5. law of contrapositive inference
6. AVT	6. given
7. A	7. law of disjunctive inference

SUMMARY:

What are the key steps in writing formal two column proofs?

HOMEWORK:

Write formal proofs.

1. $P_1: w \rightarrow x$
 $P_2: x \rightarrow r$
 $P_3: w$
concl: r

2. $P_1: s \rightarrow \sim t$
 $P_2: u \vee s$
 $P_3: t$
 $P_4: u \rightarrow v$
concl: v

3. Given the following sentences:

Either I do the geometry question or I do the algebra question.

If I do the algebra question, I get it correct.

If I get the algebra question correct, I do not lose points.

However, it is known that I lost points.

Therefore, I did the geometry question.

Let G represent "I do the geometry question."

Let A represent "I do the algebra question."

Let C represent "I get the algebra question correct."

Let P represent "I lose points."

- a) Using G , A , C , P and proper logic connectives, express each sentence in symbolic form.
- b) Using laws of inference, show that a valid conclusion has been reached.

4. Given the following sentences:

If the programmer is skilled, the computer will be accurate.

Either the programmer is skilled or the employees are lazy.

If the employees are lazy, John will be fired.

If the computer is accurate, Harry will get a raise.

John did not get fired.

Let P represent "The programmer is skilled."

Let C represent "The computer is accurate."

Let E represent "The employees are lazy."

Let J represent "John will be fired."

Let H represent "Harry will get a raise."

a) Using P, C, E, J, H and proper connectives, express each sentence in symbolic form.

b) Using laws of inference, show that Harry will get his raise.

Answers:

1.	Statements	Reasons
	1. $w \rightarrow x$	1. given
	2. $x \rightarrow r$	2. given
	3. $w \rightarrow r$	3. chain rule
	4. w	4. given
	5. r	5. law of detachment
2.	Statements	Reasons
	1. $s \rightarrow \sim t$	1. given
	2. t	2. given
	3. $\sim s$	3. contrapositive inference
	4. $u \vee s$	4. given
	5. u	5. disjunctive inference
	6. $u \rightarrow v$	6. given
	7. v	7. law of detachment

3. a) $P_1: G \vee A$
 $P_2: A \rightarrow C$
 $P_3: C \rightarrow \sim P$
 $P_4: P$
concl: G

Statements	Reasons
1. $A \rightarrow C$	1. given
2. $C \rightarrow \sim P$	2. given
3. $A \rightarrow \sim P$	3. chain rule
4. P	4. given
5. $\sim A$	5. contrapositive inference
6. $G \vee A$	6. given
7. G	7. disjunctive inference

4. $P_1: P \rightarrow C$
 $P_2: P \vee E$
 $P_3: E \rightarrow J$
 $P_4: C \rightarrow H$
 $P_5: \sim J$
concl: H

Statements	Reasons
1. $E \rightarrow J$	1. given
2. $\sim J$	2. given
3. $\sim E$	3. contrapositive inference
4. $P \rightarrow C$	4. given
5. $C \rightarrow H$	5. given
6. $P \rightarrow H$	6. chain rule
7. $P \vee E$	7. given
8. P	8. disjunctive inference
9. H	9. law of detachment

Lesson 8 - Logic

AIM: How do we use the laws of logic to write more complicated proofs? (2nd lesson)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Develop plans in order to write logic proofs.
2. Write logic proofs similar to those presented on Course II Regents exams.

VOCABULARY: plan, translate, connective

MOTIVATION:

Warm-up: Write a two column proof for the following

conclusion: $P_1: s \rightarrow t$

$P_2: \sim s \rightarrow u$

conclusion: $\sim t \rightarrow u$

DEVELOPMENT:

1. Assign a student to write his statements for the proof on the board. Elicit the reasons from the class (law of contrapositive and law of syllogism).
2. Review the steps that go into formulating a formal proof.
3. Then present the following problem to the class (Regents Exam 6/83 #42).

Given:

If Jill studies, then she will know the work.

If Jill goes out to play, then she will not know the work.

Jill studies or her parents will not be happy.

Jill's parents are happy or she will not be allowed to go to the party.

Jill is allowed to go to the party.

Prove: Jill does not go out to play.

Ask, "What is the first step?" Elicit that each statement must be expressed symbolically. Let us use:

Let S represent: "Jill studies."

Let W represent: "Jill will know the work."

Let P represent: "Jill goes out to play."
 Let H represent: "Jill's parents are happy."
 Let A represent: "Jill is allowed to go to the party."

The statements you elicit should be

Given:

- $P_1: S \rightarrow W$
 $P_2: P \rightarrow \sim W$
 $P_3: S \vee \sim H$
 $P_4: H \vee \sim A$
 $P_5: A$

Conclusion: $\sim P$

Elicit that the first step is to formulate a plan. Ask: What will allow us to conclude $\sim P$? Elicit W since from $P \rightarrow \sim W$ and the law of contrapositive inference we can conclude $\sim P$. Therefore the plan is:

$\sim P$
 $P \rightarrow \sim W$
 W

Ask: How can we get W? We must use $S \rightarrow W$ and therefore need S. (Note: By breaking up the proof into smaller proofs, the students do not feel so overwhelmed.) Now let's try to find S. Elicit that we need two columns.

Statements	Reasons
Ask: How should we begin? Elicit: With given premises.	
1. $H \vee \sim A$	1. given
2. A	2. given
What does this lead to?	
3. H	3. law of disjunctive inference
What should come next?	
4. $S \vee \sim H$	4. given
Which leads us to	
5. S	5. law of disjunctive inference
What can we do now? Elicit: Use our plan.	
6. $S \rightarrow W$	6. given
7. W	7. law of detachment
8. $P \rightarrow \sim W$	8. given
9. $\sim P$	9. law of contrapositive inference

Again review the steps followed to do a formal proof.

APPLICATIONS:

Write a formal proof for each of the following:

1. $P_1: r \rightarrow n$
 $P_2: \sim r \rightarrow m$
 $P_3: \sim n$
 concl: m

2. $P_1: \sim y$
 $P_2: x \rightarrow z$
 $P_3: (\sim y) \rightarrow (\sim z)$
 concl: $\sim x$

3. Given the sentences:

If Carmine has time, then he will play tennis.
 If the racquet string breaks, then he can't play tennis.
 The racquet string breaks or Carmine has tennis elbow.
 Carmine has time.

Prove: Carmine has tennis elbow.

4. Given the sentences:

If apples are red, then cherries are pink.
 Apples are red or Prince is a disco group.
 If dogs don't bark, then cherries aren't pink.
 Prince is not a disco group.

Prove: Dogs bark.

Answers:

1. Statements	Reasons
1) $r \rightarrow n$	1) given
2) $\sim n$	2) given
3) $\sim r$	3) contrapositive inference
4) $\sim r \rightarrow m$	4) given
5) m	5) detachment

2. Statements	Reasons
1) $(\sim y) \rightarrow (\sim z)$	1) given
2) $z \rightarrow y$	2) law of contrapositive
3) $x \rightarrow z$	3) given
4) $x \rightarrow y$	4) chain rule
5) $\sim y$	5) given
6) $\sim x$	6) contrapositive inference

3. $P_1: C \rightarrow T$	Statements	Reasons
$P_2: R \rightarrow \sim T$	1) $C \rightarrow T$	1) given
$P_3: R \vee E$	2) C	2) given
$P_4: C$	3) T	3) law of detachment
concl: E	4) $R \rightarrow \sim T$	4) given
	5) $\sim R$	5) contrapositive inference
	6) $R \vee E$	6) given
	7) E	7) disjunctive inference

4. $P_1: A \rightarrow C$
 $P_2: A \vee P$
 $P_3: (\sim D) \rightarrow (\sim C)$
 $P_4: \sim P$
concl: D

Statements	Reasons
1) $A \vee P$	1) given
2) $\sim P$	2) given
3) A	3) disjunctive inference
4) $A \rightarrow C$	4) given
5) $(\sim C) \rightarrow (\sim A)$	5) law of contrapositive
6) $(\sim D) \rightarrow (\sim C)$	6) given
7) $(\sim D) \rightarrow (\sim A)$	7) chain rule
8) D	8) contrapositive inference (from steps 3 and 7)

SUMMARY:

How do we use the laws of logic to do logic proofs?

HOMEWORK:

Write a formal proof for each of the following.

1. $P_1: p \rightarrow q$
 $P_2: r \rightarrow \sim q$
 $P_3: r$
concl: $\sim p$

2. $P_1: s \rightarrow t$
 $P_2: t \rightarrow u$
 $P_3: s \vee x$
 $P_4: x \rightarrow y$
 $P_5: \sim y$
concl: u

3. Given the sentences:

If horses fly, then 7 is not a prime number.

Either the Mets will win the pennant or horses fly.

7 is a prime number.

If the Mets win the pennant, then they will win the World Series.

Prove: The Mets win the World Series.

4. Given the sentences:

If doves fly, then let's hear it for the boys.

If doves don't fly, then I just called to say I love you.

If let's get crazy and let's hear it for the boys, then I left my heart in San Francisco.

I didn't leave my heart in San Francisco.

Let's get crazy.

Prove: I just called to say I love you.

Answers:

1.	Statements	Reasons
1)	$p \rightarrow q$	1) given
2)	$(\sim q) \rightarrow (\sim p)$	2) contrapositive
3)	$r \rightarrow (\sim q)$	3) given
4)	$r \rightarrow (\sim p)$	4) chain rule
5)	r	5) given
6)	$\sim p$	6) detachment

2.	Statements	Reasons
1)	$x \rightarrow y$	1) given
2)	$\sim y$	2) given
3)	$\sim x$	3) contrapositive inference
4)	$s \vee x$	4) given
5)	s	5) disjunctive inference
6)	$s \rightarrow t$	6) given
7)	$t \rightarrow u$	7) given
8)	$s \rightarrow u$	8) chain rule
9)	u	9) detachment (from steps 5 and 8)

3. $P_1: H \rightarrow (\sim S)$

$P_2: M \vee H$

$P_3: S$

$P_4: M \rightarrow W$

concl: W

Statements	Reasons
1) $H \rightarrow (\sim S)$	1) given
2) S	2) given
3) $\sim H$	3) contrapositive inference
4) $M \vee H$	4) given
5) M	5) disjunctive inference
6) $M \rightarrow W$	6) given
7) W	7) detachment

4. $P_1: D \rightarrow H$

$P_2: (\sim D) \rightarrow L$

$P_3: (C \wedge H) \rightarrow F$

$P_4: \sim F$

$P_5: C$

concl: L

Statements	Reasons
1) $(C \wedge H) \rightarrow F$	1) given
2) $\sim F$	2) given
3) $\sim(C \wedge H)$	3) contrapositive inference
4) $(\sim C) \vee (\sim H)$	4) DeMorgan's Law of negation
5) C	5) given
6) $\sim H$	6) disjunctive inference
7) $D \rightarrow H$	7) given
8) $(\sim H) \rightarrow (\sim D)$	8) law of contrapositive
9) $(\sim D) \rightarrow L$	9) given
10) $(\sim H) \rightarrow L$	10) chain rule
11) L	11) law of detachment (from steps 6 and 10)

Lesson 9 - Logic

AIM: What role do quantifiers play in logic?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and apply the universal quantifier ($\forall$).
2. Define and apply the existential quantifier ($\exists$) to simple sentences, including algebraic and geometric sentences.
3. Negate statements involving quantifiers verbally and symbolically.
4. Determine the truth value of original and negated sentences involving quantifiers.

VOCABULARY: universal, existential, quantifier, negate, translate

MOTIVATION:

Negate: All basketball players are taller than 7'.

DEVELOPMENT:

1. What is the key word in the motivating statement? (all)
Explain that, as with other words used in logic, there is a symbol for all: This is called the universal quantifier. If we were to use bp for basketball player, how would you express that sentence symbolically?

($\forall(bp): bp > 7'$ tall)

Let's apply this to algebraic statements: translate $\forall x$:
 $x - 5 > 0$. Translation: for all x, x - 5 is greater than 0. Ask: Is this a true statement? Why? Elicit that this is a false statement, since if $x = 3$, it is not true. Have the class write:

- a) For all numbers, a number added to zero equals the number, symbolically. Answer: $\forall x: (x+0 = x)$
- b) $\forall x: (x + 5 > 2)$, in words. Answer: For all x, x+5 is greater than 2.

2. Consider b) above: Is $\forall x: (x + 5 > 2)$ a true statement?
Answer: No, since if $x = -5$, we would have $0 > 2$, which is false.

This can be written as $\exists x: (x + 5 \leq 2)$. Ask: How can we translate this into words? Elicit: $\exists x: (x + 5 \leq 2)$ says there exists an x (a number) such that x plus 5 is less than or equal to 2. Explain that $\exists$ is called the existential quantifier.

Have the class write:

- a) There exists a rectangle that is not a square, symbolically. ($\exists$ rect: (the rectangle is not a square).
- b) $\exists x: (2x + 7 = 19)$, in words. (There exists an x such that $2x + 7$ equals 19.)

3. Elicit the answer to the motivation. (Negate: All basketball players are taller than 7'.) Accept any response that indicates there is a basketball player who is not taller than 7', i.e., Michael Jordan is 6'5" and is a basketball player. Elicit: How do we negate any statement that contains the universal quantifier or the word "all"? Answer: We show one example that contradicts the statement. This makes an "all" or $\forall x$ statement false.

Ask: Negate:

- a) $\forall x: (3x - 5 > 0)$ Ans: $\exists x: (3x - 5 \leq 0)$, (if $x = 1$),
- b) All reindeer have red noses. Ans: There is a reindeer without a red nose. or: Some reindeer don't have red noses.

4. Negate: Some squares are not rectangles.

Answer: All squares are rectangles. or: No squares are not rectangles.

Ask: How do we negate a statement containing the existential ($\exists x$) quantifier or the word "some"? Elicit two ways and write them on the chalkboard. 1) Change "some" to "no" or "none" and keep the sentence as is. 2) Change "some" to "all" and negate the remainder of the sentence.

5. Determine the truth value of each and negate.
- a) $\forall x: (x + 0 = x)$ b) $\exists x: (x + 3 < 4)$
 c) All Mondays are holidays.

Elicit the truth values and the negations.

- a) $\exists x: (x + 0 \neq x)$
 b) $\exists x: (x + 3 \geq 4)$
 c) Some Mondays are not holidays.

APPLICATIONS:

1. Translate each of the following into words.

a) $\exists x: (x + 7 > 5)$

Ans: There exists an x such that x plus 7 is greater than 5.

b) $\forall y: (2y + 3 = 9)$

Ans: For all y , 2 times y plus 3 equals 9.

c) $\exists z: (2z + z < 9)$

Ans: There exists a z such that twice z plus z is less than 9.

2. Translate into symbols using quantifiers.

a) For every number, the sum of the number and zero is the number. Ans: $\forall x: (x + 0 = x)$

b) There is a number that when doubled and added to 7 is equal to 13. Ans: $\exists x: (2x + 7 = 13)$

3. Determine the truth value of each of the following and negate it.

a) All cows give chocolate milk. Ans: False

Negation: Some cows don't give chocolate milk.

b) Some triangles are round. Ans: False

Negation: All triangles are not round or No triangles are round.

c) Some people work on Sunday. Ans: True

Negation: All people don't work on Sunday or No people work on Sunday.

d) All good students do their homework. Ans: True

Negation: Some good students don't do their homework.

SUMMARY:

1. How do we negate a statement containing $\forall x$?
2. Name two ways to negate: There is a weeble that wiggles.

HOMEWORK:

1. Translate into words.
 - a) $\forall x: (x \cdot 1 = x)$
Ans: For all x , x times 1 is x .
 - b) $\exists y: (y + 0 > y)$
Ans: There exists a y such that y plus 0 is greater than y .
2. Determine the truth value and negate each of the following.
 - a) $\forall x: (2x = x)$
Ans: False. Neg: $\exists x: (2x \neq x)$
 - b) $\exists y: (y + 7 > 6)$
Ans: True. Neg: $\sim \exists y: (y + 7 > 6)$ or $\forall y: (y + 7 \leq 6)$
 - c) Some rectangles are squares.
Ans: True. Neg: All rectangles are not squares.
or: No rectangles are squares.
 - d) All horses are thoroughbreds.
Ans: False. Neg: Some horses are not thoroughbreds.
 - e) Some dancers are not pretty.
Ans: True. Neg: All dancers are pretty.
or No dancers are not pretty.

Lesson 10 - Proofs Involving Lines and Angles

AIM: What are the foundations of an axiomatic system?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Distinguish between undefined and defined terms.
2. Distinguish between axioms and theorems.
3. Give the requirements of a good definition.
4. Explain the difference between inductive and deductive reasoning.

VOCABULARY: axiom, postulate, theorem, deductive, inductive

MOTIVATION:

What is a point? What is a triangle?

Elicit possible responses:

Point: place, location, etc.

Triangle: 3-sided closed figure, shape

DEVELOPMENT:

Ask: What is a "place"? Elicit that a description of "point" may involve "place", but that if "place" is described as "point", you can't learn from either description unless you already know what each term means. We therefore must accept some terms as undefined, since a good definition should:

State the exact meaning of a term by stating the class the object belongs in and the feature which separates it from others in the class. A good definition should use only terms already defined or accepted without definition.

"Point" and "line" are undefined, although we accept certain descriptions of each.

Point: denotes position in space, with no length or width or depth (height).

Line: set of points extending indefinitely in two opposite directions.

Why is this a poor definition: A triangle is a geometric figure? (It is not distinguished from square, rectangle, etc.) Under what circumstances would this be a good definition: A triangle is a closed figure with three sides? (Only if "figure" and "side" are already defined or accepted without definition.) A definition should be reversible (a closed figure with three sides is a triangle).

In geometry we define and develop properties of various geometric figures. Some of these properties are proved and others are accepted without proof. A theorem is a property that is proved. An axiom (or postulate) is a property accepted without proof.

Properties can be suggested by continuous observation (induction) or by logical reasoning and previously known properties (deduction). Claiming that the measures of the angles in every triangle total 180° because the sum of the measures of the angles of every triangle we inspected is 180° is an example of INDUCTIVE reasoning. Making the same claim on the basis of other properties and rules of logic is DEDUCTIVE reasoning. Most theorems are suggested by inductive reasoning but proved by deductive reasoning. A body of knowledge based on undefined terms, definitions, axioms and theorems is called an AXIOMATIC system.

APPLICATIONS:

1. Is "number" defined or undefined? If defined, write a good definition. If undefined, write a description. (Most definitions use words so closely related to number that a "good" definition is hard to write - leave it undefined.)
2. Write a good definition for "even number" (a number which is the double of some integer).
3. Criticize this definition of pencil: "A pencil is a tool used in writing." (It is not specific enough; when you reverse it - a tool used in writing is a pencil - you have a false statement.)

4. Are DeMorgan's Laws (in logic) axioms or theorems?
Explain. (Theorems; we proved them.)
5. If a baby touches a hot stove, what kind of reasoning might protect it from touching another hot stove?
(Inductive; from an example, the baby concludes that hot stoves are painful.)

SUMMARY:

Ask: What are some undefined terms in geometry? What is the difference between an axiom and a theorem?

HOMEWORK:

1. Write a good definition for
 - a) "odd number"
 - b) "prime number"(Assume "number" has been defined or accepted as undefined.)
2. In Course I we said that in a right triangle $a^2 + b^2 = c^2$. Is this an axiom or theorem?
3. Find a dictionary definition of "line". Does it meet our requirements of a good definition? Why?
4. In solving algebraic equations, do we use inductive or deductive reasoning?

Lesson 11 - Proofs Involving Lines and Angles

AIM: What are the basic undefined and defined terms concerning lines?

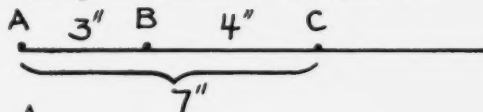
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define collinear points, betweenness of points on a line.
2. Define congruent line segments and midpoint of a line segment.
3. Define the tools of a geometric construction.
4. Construct a line segment congruent to a given segment.
5. Bisect a given line segment.

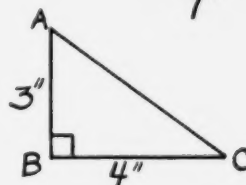
VOCABULARY: collinear, between, congruent segments, midpoint

MOTIVATION:

Pose the question, "If point A is 3" from point B, which is 4" from point C, what is the number of inches from A to C?" If students respond 7", draw the diagram below and ask if any other possibilities exist.



Consider the case where $\angle ABC$ is a right angle. Elicit the Pythagorean Theorem (from Course I) and the fact that A may be 5" from C.

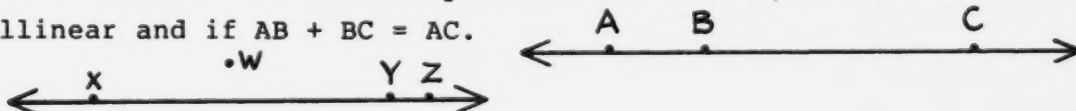


How does the second example differ from the first? (A is not on the same line as B and C.)

DEVELOPMENT:

Define: **COLLINEAR POINTS** are points which all lie on the same line. Remind students that $\overline{AB}$ identifies a line segment and AB the length of that segment (from Course I).

Define: Point B is **BETWEEN** points A and C if A, B and C are collinear and if $AB + BC = AC$.



In the previous diagram, is W between X and Y? Is Y between X and Z? Is X between Y and Z? Justify each answer. (Only Y is between X and Z since the three points are collinear and $XY + YZ = XZ$.)

Remind students that congruent ($\cong$) segments have equal lengths and that the midpoint of a segment divides the segment into two congruent segments. In the design or manufacture of various items it is necessary to duplicate a line segment or bisect a segment (divide it into two congruent segments). Because we often cannot measure the exact length of a segment with a ruler, we will use a compass and straight edge for our constructions.

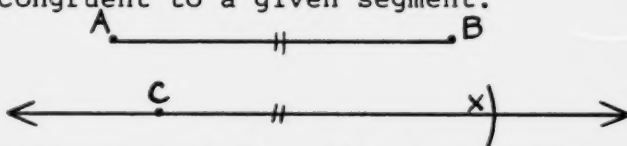
1. Constructing a segment congruent to a given segment.

Given segment $\overline{AB}$.

Draw any line $\overleftrightarrow{CD}$.

Place the point of the

compass at A and the pencil point at B. Without changing the compass setting, place the steel point at C and construct an arc intersecting $\overleftrightarrow{CD}$ at a point X. Now $\overline{CX} = \overline{AB}$.



2. Bisecting a given segment.

Given segment $\overline{AB}$. Open the compass so that the distance

between the two compass points

(call this the "radius") is more than half the length of

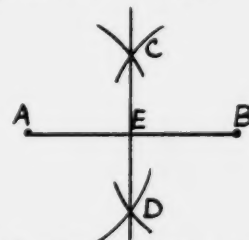
$\overline{AB}$. Place the steel point at A and construct an arc

above $\overline{AB}$ and an arc below $\overline{AB}$. Place the steel point at B

and construct another set of arcs above and below $\overline{AB}$

which intersect the first arcs at C and D. Construct

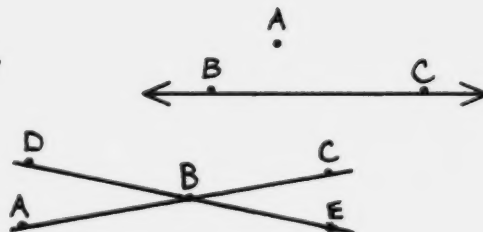
$\overleftrightarrow{CD}$ intersecting $\overline{AB}$ at E. E is the midpoint of $\overline{AB}$ and $\overline{CD}$ bisects $\overline{AB}$.



APPLICATIONS:

1. Why is A not between B and C?
(not collinear)

2. Name three collinear points.
(A, B, C or D, B, E)



3. Construct a segment congruent to $\overline{LM}$.

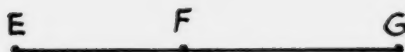


4. Construct the bisector of $\overline{XY}$.



5. If F is between E and G:

- find EG if EF = 4 and FG = 5. (9)
- find FG if EG = 8 and EF = 3.5. (4.5)
- find EG if EF = 5 and F bisects $\overline{EG}$. (10)



SUMMARY:

Ask: How are collinearity and betweenness related? (Points must be collinear for one to be between two others.)

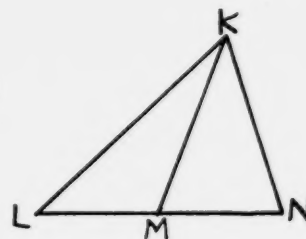
Ask the class to describe, in their own words, how to construct a segment congruent to a given segment and how to bisect a given segment.

HOMEWORK:

1. In fig. 1, $\overleftrightarrow{LMN}$.

(fig. 1)

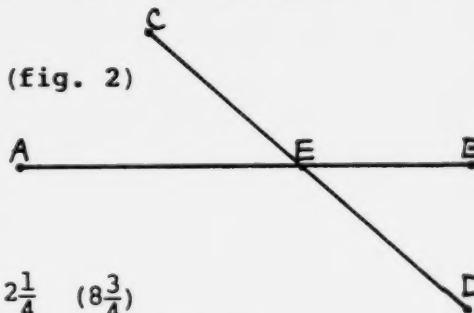
- Name 3 collinear points (L, M, N)
- Name 3 non-collinear points (L, M, K or M, K, N, etc.)
- What statement concerning betweenness can be made? (M is between L and N.)
- If $\overline{KM}$ bisects $\overline{LN}$, name two congruent segments. ($\overline{LM} \cong \overline{MN}$)



2. In fig. 2, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$.

(fig. 2)

- Find CD if CE = 8, ED = 3 (11)
- Find AE if AB = 12, EB = 6 (6)
- Find CE if CD = 11, ED = $2\frac{1}{4}$ ($8\frac{3}{4}$)
- Find x if ED = x, CE = 8 and CD = 12 (4)



3. Using fig. 2, $\overleftrightarrow{CD}$, $\overleftrightarrow{AB}$, solve for x if:

a) $ED = x$, $CE = 3x$, $CD = 16$ (4)

b) $CD = 11$, $CE = x + 2$, $ED = x + 3$ (3)

c) $AE = 4x$, $EB = x - 7$, $AB = 2x + 35$ (14)

4. Construct a segment congruent to $\overline{AB}$ in fig. 2.

5. Construct the bisector of $\overline{LM}$. 

6. Construct a segment twice the length of $\overline{AB}$ in fig. 2.

7. Construct a segment one quarter the length of $\overline{XY}$.



Lesson 12 - Proofs Involving Lines and Angles

AIM: What is the basic terminology of angles and angle measurement?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define ray, angle, congruent angles, bisector of an angle.
2. Define acute, right, obtuse, straight and reflex angles.
3. Construct an angle congruent to a given angle.
4. Bisect a given angle.

VOCABULARY: ray, angle, congruent angles, bisector of an angle, acute, right, obtuse, straight and reflex angles

MOTIVATION:

Ask, "How does a television set receive a signal beamed in by satellite?" (An antenna shaped like a large dish is aimed at the satellite by pointing it in the proper direction and at the proper angle.)

How can we properly define an angle?

DEVELOPMENT:

Define: RAY is a part of a line consisting of a point on the line (an endpoint) and all the points on one side of the line.

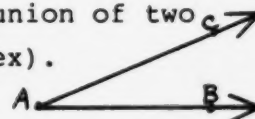
$\overrightarrow{BA}$ and $\overrightarrow{BC}$ are rays.



How can we use ray in defining angle?

Define: ANGLE is a set of points that is the union of two rays having the same endpoint (called the vertex).

(Diagram shows $\angle CAB$ or $\angle A$.)



interior of $\angle CAB$

Elicit (based on work in Course I) definitions of acute, right, obtuse, straight and reflex angles (angles measuring less than 90° , 90° , more than 90° but less than 180° , 180° , more than 180°). Draw a representative angle from each category on the chalkboard.

Ask: Based on our definitions concerning segments, how would you define congruent angles and angle bisector?

Define: CONGRUENT ANGLES are angles which have the same measure.

Define: ANGLE BISECTOR is a ray whose endpoint is the vertex of the angle and which divides the angle into two congruent angles.

$$\angle ABC \cong \angle XYZ$$

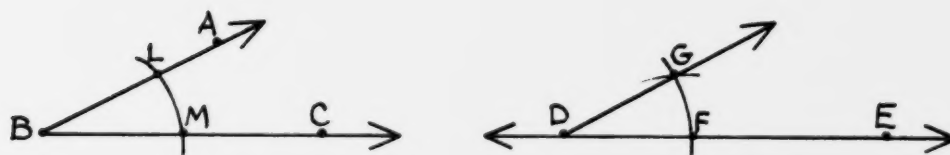
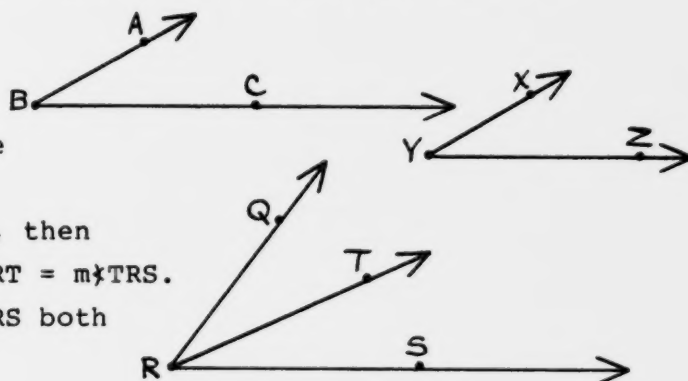
$$m\angle ABC = m\angle XYZ$$

($m\angle ABC$ indicates the measure of $\angle ABC$.)

If $\overrightarrow{RT}$ bisects $\angle QRS$, then

$$\angle QRT \cong \angle TRS \text{ and } m\angle QRT = m\angle TRS.$$

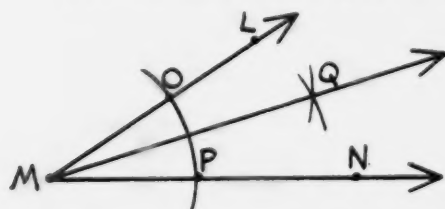
$$\text{Also, } m\angle QRT \text{ and } m\angle TRS \text{ both} \\ = \frac{1}{2}m\angle QRS.$$



I. To construct an angle congruent to $\angle ABC$ at point D.

1. Through point D, draw line $\overleftrightarrow{DE}$.
2. Placing the steel compass point at B, construct an arc intersecting $\overrightarrow{BA}$ and $\overrightarrow{BC}$ at L and M, respectively.
3. Placing the steel point at D, construct an arc with the same radius intersecting $\overleftrightarrow{DE}$ at F.
4. Reset the radius so one point of the compass is on M and the other on L.
5. Using this radius, place the steel point at F and construct an arc intersecting the previous arc at G.
6. Construct $\overrightarrow{DG}$. Now, $\angle GDE \cong \angle ABC$.

II. To bisect a given angle:



1. Placing the point at M, construct an arc intersecting $\overrightarrow{ML}$ and $\overrightarrow{MN}$ at O and P, respectively.

2. With centers of O and P (placing the point at O and then at P), construct arcs of equal radii in the interior of $\angle LMN$.
3. Label the intersection of these arcs Q. $\overrightarrow{MQ}$ bisects $\angle LMN$.

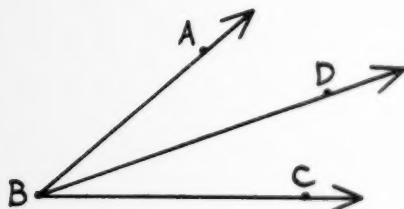
APPLICATIONS:

1.



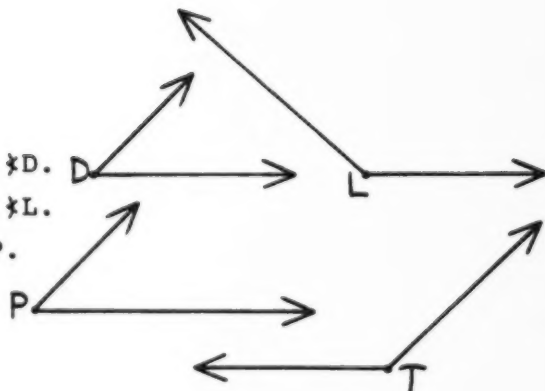
If $\angle LMN \cong \angle OPQ$, find the value of x . (20°)

2.



- a) What value of x results from $\overrightarrow{DB}$ bisecting $\angle ABC$? (5°)
- b) What must be the measure of $\angle ABC$, $\angle ABD$ and $\angle DBC$? ($60^\circ, 30^\circ, 30^\circ$)

3. Construct an angle congruent to $\angle D$.
4. Construct an angle congruent to $\angle L$.
5. Construct a ray which bisects $\angle P$.
6. Construct the bisector of $\angle T$.



SUMMARY:

Ask: When is an angle congruent to another angle? (When their measures are equal.) How can we bisect a given angle? (see development)

HOMEWORK:

1. Name 6 angles with P as vertex. ($\angle APB$, $\angle BPC$, $\angle CPD$, $\angle APC$, $\angle BPD$, $\angle APD$)

(fig. 1-4)

2. Which angles are $\cong$ if $\overrightarrow{PC}$ bisects $\angle BPD$? ($\angle BPC$ and $\angle CPD$)

3. If $m\angle BPC = 35$ and $m\angle CPD = 3x - 7$, what value of x makes $\angle BPC = \angle CPD$? (14)

4. If $m\angle BPC = 6x + 12$ and $m\angle CPD = 40 - x$ and $\overrightarrow{PC}$ bisects $\angle BPD$, find the value of x and $m\angle BPC$, $m\angle CPD$ and $m\angle BPD$. (4, 36, 36, 72)

5. Classify the angles having each of the following measures as acute, right, obtuse, straight or reflex.

- | | |
|----------------|----------------|
| a) 90° | d) 105° |
| b) 180° | e) 185° |
| c) 15° | f) 75° |

6. Construct an angle congruent to $\angle A$.

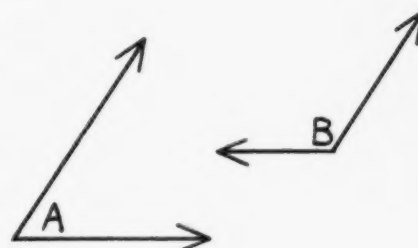
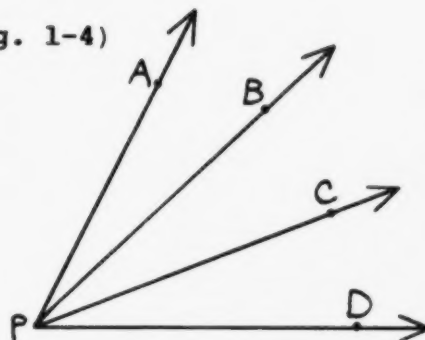
7. Construct the angle bisector of $\angle A$.

8. Construct an angle congruent to $\angle B$.

9. Construct the angle bisector of $\angle B$.

10. Construct an angle whose measure is $\frac{1}{4}$ that of $\angle B$.

11. Construct an angle whose measure is twice that of $\angle A$.



Lesson 13 - Proofs Involving Lines and Angles

AIM: What are the different types of angle pairs?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define adjacent angles.
2. Define vertical angles.
3. Define complementary angles and supplementary angles.
4. Define perpendicular lines.
5. Solve problems involving angle pairs.

VOCABULARY: adjacent angles, vertical angles, complementary angles, supplementary angles, perpendicular lines

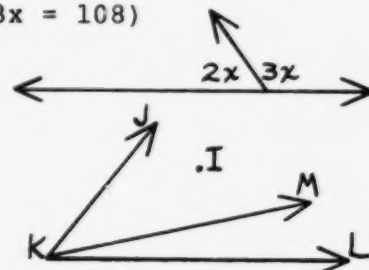
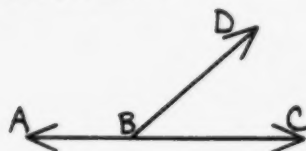
MOTIVATION:

Challenge: A straight angle is divided into two smaller angles whose measures are in ratio 2:3. Find these angle measures.

Elicit that the diagram may look like this:

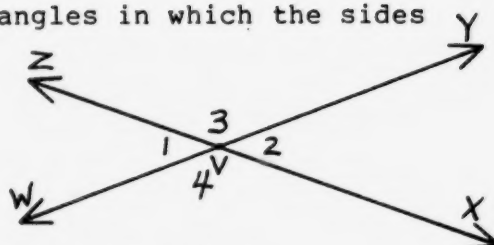
What is the measure of a straight angle? (180) What equation can we write? ($2x + 3x = 180$) How can we solve the equation? ($5x = 180$, $x = 36$, $2x = 72$, $3x = 108$)

DEVELOPMENT:



To solve problems involving angles we need terms to express ways in which angles may be related. Define: **ADJACENT ANGLES** are two angles (in the same plane) with a common vertex and a common side, but no common interior points. In the diagrams above, $\angle ABD$ and $\angle DBC$ are adjacent. So are $\angle EFH$ and $\angle HFG$. Are $\angle JKM$ and $\angle MKL$? (yes) Are $\angle JKM$ and $\angle JKL$? (no, because point I is in the interior of both angles.)

Define: VERTICAL ANGLES are two angles in which the sides of one angle are opposite rays to the sides of the other. If $\overleftrightarrow{ZX}$ and $\overleftrightarrow{WY}$ are lines then $\angle 1$ and $\angle 2$ are vertical angles and so are $\angle 3$ and $\angle 4$.

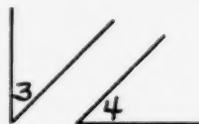


(Based on Course I work) Ask the class to define COMPLEMENTARY and SUPPLEMENTARY ANGLES.

Define: Complementary angles are two angles the sum of whose measures is 90. Each is the COMPLEMENT of the other.

Define: Supplementary angles are two angles the sum of whose measures is 180. Each is the SUPPLEMENT of the other.

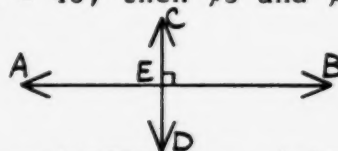
In the diagrams on the preceding page, if $\angle ABC$ is a straight angle, then $\angle ABD$ and $\angle DBC$ are supplements. If $\angle EFG$ is a right angle, then $\angle EFH$ and $\angle HFG$ are complements. Note: Angles need not be adjacent to be complementary or supplementary.



For example, if $m\angle 1 = 160$ and $m\angle 2 = 20$, $\angle 1$ and $\angle 2$ are supplementary. If $m\angle 3 = 42$ and $m\angle 4 = 48$, then $\angle 3$ and $\angle 4$ are complementary.

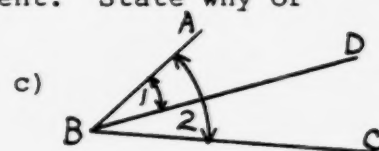
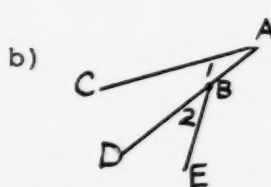
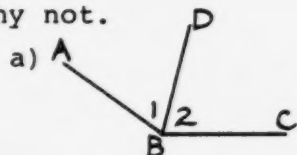
Define: PERPENDICULAR LINES are lines which intersect at right angles.

If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines and $\angle CEB$ is a right angle, then $\overleftrightarrow{CD}$ is perpendicular to $\overleftrightarrow{AB}$. ($\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$) We can also say that $\overline{AB} \perp \overline{CD}$.



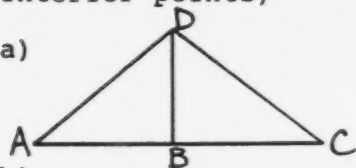
APPLICATIONS:

- In a-c determine if $\angle 1$ and $\angle 2$ are adjacent. State why or why not.

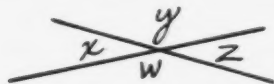


(a - yes; b - no, no common vertex; c - no, some common interior points)

2. a)



b)



Name 2 pairs of adjacent angles ($\angle ABD$, $\angle DBC$)

($\angle ADB$, $\angle BDC$)

Name 2 pairs of vertical angles ($\angle X$, $\angle Z$)

($\angle Y$, $\angle W$)

3. Find the measure of the complement of each angle.

Answers

- a) 30°
- b) 55°
- c) $14^\circ 30'$
- d) x
- e) $90 - x$

- a) 60°
- b) 35°
- c) $75^\circ 30'$
- d) $90 - x$
- e) x

4. Find the measure of the supplement of each angle.

Answers

- a) 70°
- b) 45°
- c) $68^\circ 15'$
- d) 90°
- e) 98°
- f) x
- g) $180 - x$

- a) 110°
- b) 135°
- c) $111^\circ 45'$
- d) 90°
- e) 82°
- f) $180 - x$
- g) x

5. Find two complementary angles in which the measure of the larger is 10 more than that of the smaller.

$$(x + x + 10 = 90, x = 40, x + 10 = 50)$$

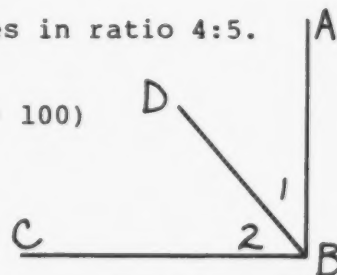
6. Which angle is equal in measure to its supplement? (a right angle; measure 90)

7. Two supplementary angles have measures in ratio 4:5. Find their measures.

$$(4x + 5x = 180, x = 20, 4x = 80, 5x = 100)$$

8. If $\overline{AB} \perp \overline{BC}$, $m\angle 1 = 4x + 18$ and $m\angle 2 = 3x - 12$, find x and the measure of each angle.

$$(4x + 18 + 3x - 12 = 90, 7x + 6 = 90, x = 12, m\angle 1 = 66, m\angle 2 = 24)$$



9. Recall from Course I that the sum of the measures of the angles of a triangle is 180. What conclusion can be drawn about the acute angles of any right triangle? (they are complementary)

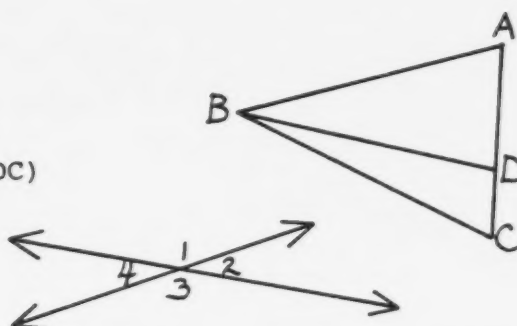
SUMMARY:

Ask the class to explain the meanings of adjacent, complementary, supplementary and vertical angle pairs.

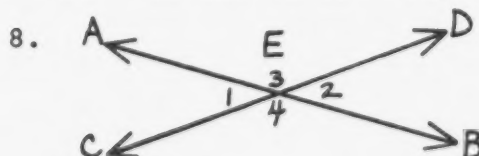
HOMEWORK:

1. Name two pairs of adjacent angles.
($\angle ABD$, $\angle DBC$) ($\angle ADB$, $\angle BDC$)

2. Name two pairs of vertical angles.
($\angle 1$, $\angle 3$) ($\angle 2$, $\angle 4$)



3. Find the measure of the complement and the supplement of each angle.
- | | | | |
|-------------------|--------------------------------------|-------------|----------------------|
| a) 40° | (50° , 140°) | d) a | ($90-a$, $180-a$) |
| b) 65° | (25° , 115°) | e) $90 - t$ | (t , $90+t$) |
| c) $15^\circ 20'$ | ($74^\circ 40'$, $164^\circ 40'$) | | |
4. Find the measure of an angle if it is 18 more than the measure of its complement. (54)
5. The measures of two supplementary angles are in ratio 2:7. Find each measure. (40, 140)
6. The measure of the supplement of an angle is 50 more than twice the measure of the angle. Find the measure of the angle. ($x+2x+50 = 180$, $3x = 130$, $x = 43\frac{1}{3}$)
7. The non-common sides of two adjacent angles are perpendicular. If the measure of one angle is represented by $2x$ and the other by $3x - 20$, find the measure of each angle. ($2x+3x-20 = 90$, $5x = 110$, $x = 22$; $2x = 44$ and $3x-20 = 46$)



$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E.
 $m\angle 1 = 5x - 10$, $m\angle 2 = 3x + 20$.
Find $m\angle 1$, $m\angle 2$, $m\angle 3$, $m\angle 4$.

$(5x-10 = 3x+20, x = 15; m\angle 1 = m\angle 2 = 65; m\angle 3 = m\angle 4 = 115$
since $\angle 3$ is the supplement of $\angle 1$)

9. $\triangle ABC$ is a right triangle with the right angle at C. $m\angle A = 8x+4$ and $m\angle B = 5x-5$. Find the measures of $\angle A$ and $\angle B$.
($8x+4+5x-5 = 90$ since the acute angles of a right triangle are complementary; $x = 7, m\angle A = 60, m\angle B = 30$)

Lesson 14 - Proofs Involving Lines and Angles

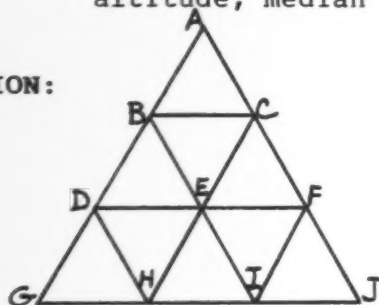
AIM: What are the special line segments associated with a triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Classify triangles according to their
 - a) sides
 - b) angles
2. Define special line segments associated with a triangle.
 - a) an altitude
 - b) a median
 - c) an angle bisector

VOCABULARY: (scalene, isosceles, equilateral - from Course I)
altitude, median

MOTIVATION:



How many distinct triangles can be seen in the adjacent diagram? (9 "small" triangles, plus $\triangle ADF$, $\triangle BGI$, $\triangle CHJ$, $\triangle AGJ$ - 13 in all.)

Since triangles are the simplest closed figures consisting of line segments, we will discuss the special line segments associated with triangles.

DEVELOPMENT:

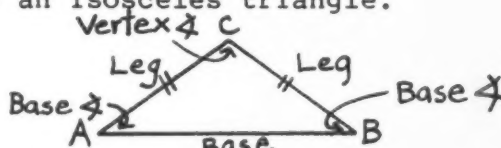
We sometimes classify triangles by how many congruent sides they have. What do we call triangles with 3 congruent sides? (equilateral) Elicit from the class that all 13 triangles in the diagram above are equilateral. (Students may recall these definitions from Course I.) With two congruent sides? (isosceles) With no congruent sides? (scalene) Write these definitions on the board as needed. Ask students to identify the parts of an isosceles triangle. If $\overline{CA} \cong \overline{CB}$:

Congruent sides are legs ($\overline{CA}$, $\overline{CB}$).

Third side is base ($\overline{AB}$).

Angle formed by congruent sides is vertex angle ($\angle C$).

Angles opposite each leg are base angles ($\angle A$, $\angle B$).



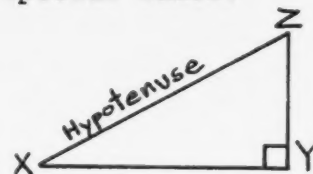
Triangles can also be classified by angles. Right triangles and obtuse triangles contain one right angle and one obtuse angle, respectively, and each contains 2 acute angles.

Acute triangles contain 3 acute angles. What is an equiangular triangle? (a triangle with 3 congruent angles)

The sides of a right triangle also have special names.

Hypotenuse is opposite the right angle ($\overline{XZ}$).

Legs are sides that form the right angle ($\overline{XY}$, $\overline{YZ}$).



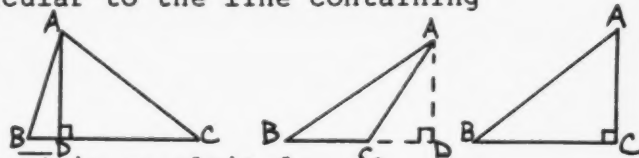
Finally we define line segments which relate "midpoint", "bisector" and "perpendicular" to triangles.

Define: ALTITUDE of a triangle is a segment drawn from any vertex of a triangle perpendicular to the line containing the opposite side.

In the first two triangles

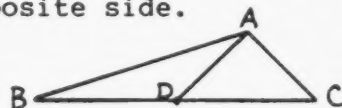
$\overline{AD} \perp \overline{BC}$ so $\overline{AD}$ is an altitude.

In the third, side $\overline{AC} \perp \overline{BC}$, so $\overline{AC}$ is an altitude.

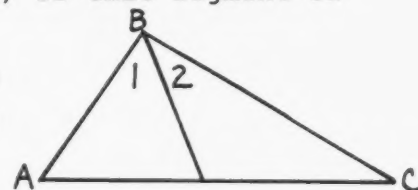


Define: MEDIAN is a segment which connects a vertex of a triangle to the midpoint of the opposite side.

If $\overline{AD}$ is a median, then D is the midpoint of $\overline{BC}$ and $\overline{BD} \cong \overline{CD}$.

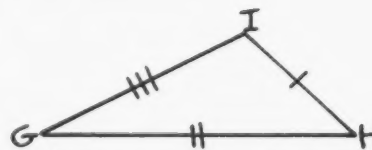
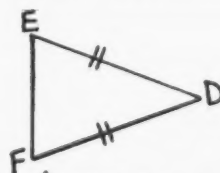
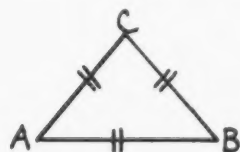


Define: ANGLE BISECTOR (in a triangle) is that segment of the angle bisector of an angle of a triangle which extends to the opposite side. $\overline{BD}$ is an angle bisector (of $\angle B$) if $\angle 1 \cong \angle 2$.



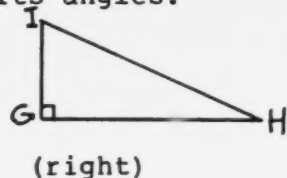
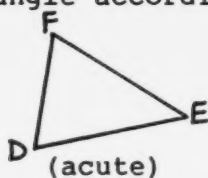
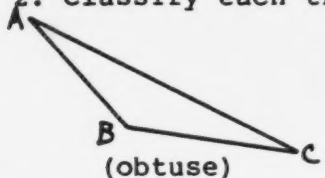
APPLICATIONS:

1. Classify each triangle according to its sides.

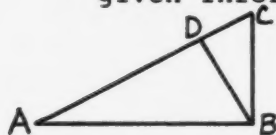


($\triangle ABC$ equilateral, $\triangle DEF$ isosceles, $\triangle GHI$ scalene)

2. Classify each triangle according to its angles.



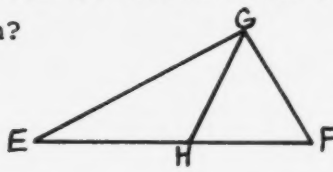
3. What conclusions can be drawn from each diagram and the given information?



given $\overline{BD}$ is an altitude

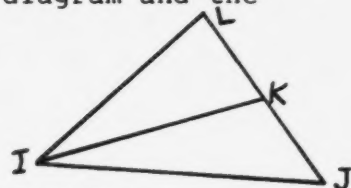
Conclusions:

($\angle BDA$, $\angle BDC$ are right $\angle$'s)



given $\overline{GH}$ is an angle bisector

($\angle EGH \cong \angle HGF$)



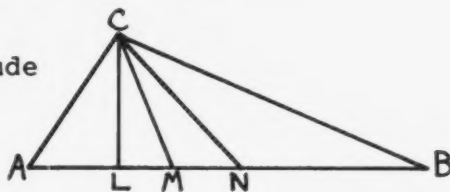
given $\overline{IK}$ is a median

($\overline{LK} \cong \overline{KJ}$)

4. Identify a median, an altitude and an angle bisector if

$\overline{AN} \cong \overline{BN}$, $\angle ACM \cong \angle BCM$, $\angle ALC$ is a right angle.

($\overline{CN}$ median, $\overline{CM}$ $\angle$ bisector, $\overline{CL}$ altitude)

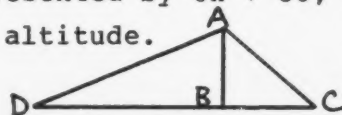


5. In isosceles triangle ABC, the length of leg $\overline{AB}$ (AB) is represented by $2x + 10$, and the length of leg $\overline{BC}$ (BC) is represented by $4x - 18$. How long is each leg?

($2x + 10 = 4x - 18 \rightarrow x = 14$, each leg = 38)

6. If the measure of $\angle ABC$ ($m\angle ABC$) is represented by $6x + 30$, find the value of x which makes $\overline{AB}$ an altitude.

($6x + 30 = 90$, $x = 10$)



7. Recall from Course I that the sum of the measures of the angles of a triangle is 180 and that the base angles of an isosceles triangle are congruent. What is the measure of a base angle of an isosceles right triangle? (45)

8. What is the measure of an angle of an equiangular triangle? (60)

SUMMARY:

Ask the class, "How can triangles be classified? How are 'midpoint' and 'perpendicular' related to 'median' and 'altitude'?"

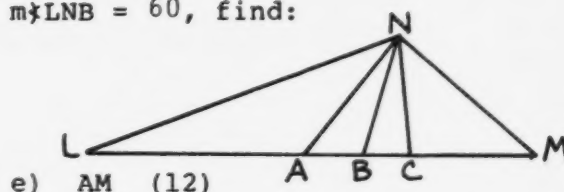
HOMEWORK:

1. Draw a triangle in each of the following categories:

- | | |
|----------------|-----------|
| a) scalene | d) acute |
| b) isosceles | e) right |
| c) equilateral | f) obtuse |

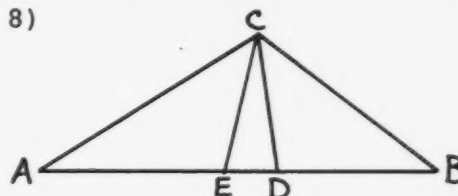
2. Given that $\overline{NB}$ is an angle bisector, $\overline{NC}$ an altitude, and $\overline{NA}$ a median, with $LA = 12$ and $m\angle LNB = 60$, find:

- a) $m\angle MNB$ (60)
 b) $m\angle LNM$ (120)
 c) $m\angle NCL$ (90)
 d) LM (24)



3. Given $m\angle LCN = 5t + 30$, $m\angle LNB = 3u + 16$, $m\angle MNB = 5u - 24$, $LA = 116 - 2v$, $AM = 20 + 10v$. Find the values of t , u and v which make $\overline{NC}$ an altitude, $\overline{NB}$ an angle bisector and $\overline{NA}$ a median. ($t = 12$, $u = 20$, $v = 8$)

4. Given: $m\angle A = 40$, $m\angle B = 50$,
 $\overline{CD}$ is an altitude,
 $\overline{CE}$ bisects $\angle ACB$



Note: $\angle ACB = 90^\circ$

- a) What name can be applied to $\angle A$ and $\angle B$?
 (complementary)
 b) What is the $m\angle ACB$? What kind of triangle is $\triangle ABC$?
 (90, right triangle)
 c) What is $m\angle ACE$? (45)
 d) What is $m\angle AEC$? (95)
 e) What is $m\angle CEB$? (85)
 f) What names can be applied to the pair $\angle AEC$ and $\angle CEB$?
 (adjacent, supplementary)
 g) What is $m\angle BCD$? (40)
 h) What is $m\angle ECD$? (5)

Lesson 15 - Proofs Involving Lines and Angles

AIM: How do we construct the special line segments associated with a triangle?

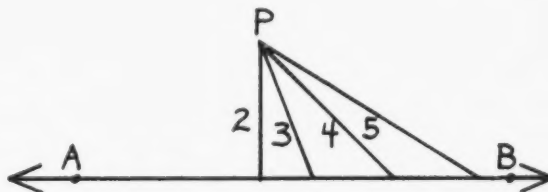
PERFORMANCE OBJECTIVES: The student will be able to...

1. Construct a line perpendicular to a given line segment through a point
 - a) on the line.
 - b) outside the line.
2. Construct the
 - a) altitude to a given side of a triangle.
 - b) median to a given side of a triangle.

VOCABULARY: no new terms

MOTIVATION:

Challenge: Find, by measuring, how far point P is from line $\overleftrightarrow{AB}$.



Ask: "How can we show the distance in the diagram? (with a segment from P to $\overleftrightarrow{AB}$) Which segment has a length equal to the distance?" (the one which is perpendicular to $\overleftrightarrow{AB}$)

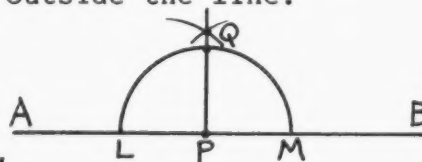
Emphasize that the distance from a point to a line is measured along a perpendicular segment.

DEVELOPMENT:

We can construct a perpendicular to a given line either at a point on the line or through a point outside the line.

CASE I (P is on $\overleftrightarrow{AB}$)

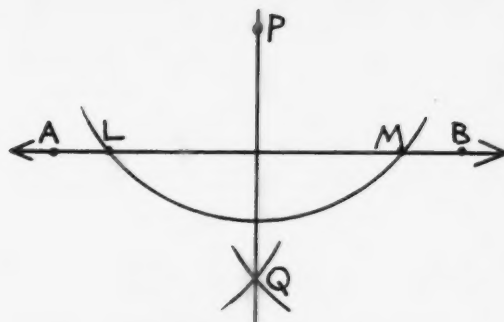
- 1) With P as the center, construct an arc intersecting $\overleftrightarrow{AB}$ at L and M.
- 2) With a larger radius, construct arcs using L and M as centers. The intersection of these arcs is Q, a point on the required line.
- 3) $\overline{QP} \perp \overline{AB}$



CASE II (P is outside $\overleftrightarrow{AB}$)

Ask, "Where shall we begin this construction?" Elicit:

- 1) Use P as a center for an arc to intersect $\overleftrightarrow{AB}$ at 2 points, L, M.
- 2) Use each of L and M as centers, constructing arcs of equal radii below $\overleftrightarrow{AB}$. These arcs intersect at Q, with $\overline{PQ} \perp \overline{AB}$.



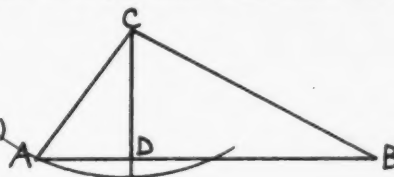
APPLICATIONS:

Ask: How can we apply our recent constructions to construct:

- a) an altitude to a given side of a triangle?
- b) a median to a given side of a triangle?

(For altitude, construct a perpendicular from the opposite vertex. For median, bisect the side first and connect the opposite vertex to the midpoint.)

1. Construct the altitude to $\overline{AB}$. ($\overline{CD}$ is the altitude to $\overline{AB}$.)

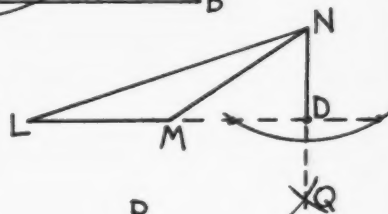


2. Construct the altitude to $\overline{LM}$.

Ask: What must be done first?

(extend $\overleftrightarrow{LM}$)

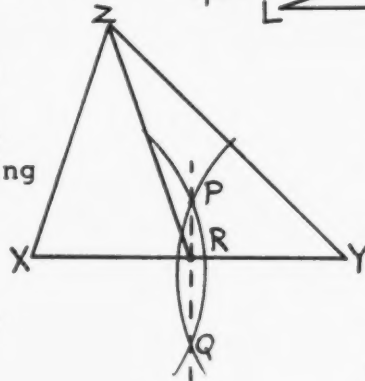
($\overline{ND}$ is the altitude to $\overline{LM}$.)



3. Construct the median to $\overline{XY}$.

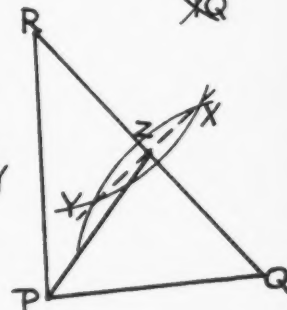
Elicit the steps in bisecting $\overline{XY}$. $\overline{PQ}$ bisects $\overline{XY}$.

($\overline{ZR}$ is the median to $\overline{XY}$.)



4. Construct the median to $\overline{RQ}$.

($\overline{PZ}$ is the median.)



SUMMARY:

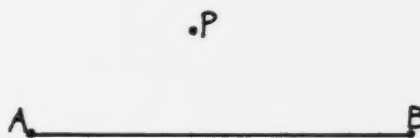
Call students to the board to construct a perpendicular

- a) at a point on a line.
- b) from a point outside the line.

Ask: How can we construct an altitude and a median to a side of a triangle? (Construct a perpendicular to the side, and then construct a bisector of the side.)

HOMEWORK:

1. Construct a line through P, perpendicular to $\overline{AB}$.
2. Construct a perpendicular to $\overline{AB}$ at P.

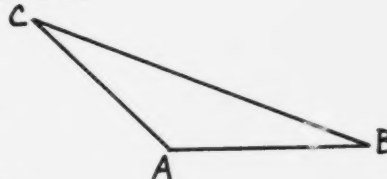


In 3 and 4 construct an altitude to $\overline{AB}$.

3.

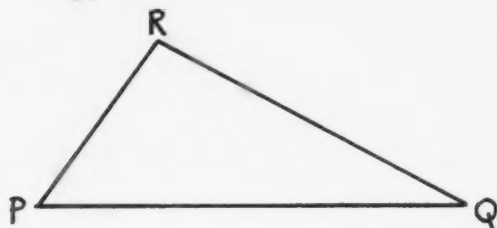


4.

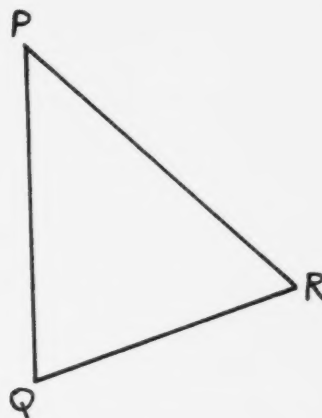


In 5 and 6 construct a median to $\overline{PQ}$.

5.



6.



Lesson 16 - Proofs Involving Lines and Angles

AIM: What is an Equivalence Relation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define the:
 - a) Reflexive Property
 - b) Symmetric Property
 - c) Transitive Property
2. Apply the above three properties to equality, congruence of line segments and congruence of angles.
3. Define Equivalence Relation. "Relation" is left undefined.
4. Test to see if a relation is an Equivalence Relation.

VOCABULARY: reflexive property, symmetric property, transitive property, equivalence relation

MOTIVATION:

In our school's volleyball intra-mural tournament, the Slammers defeated the Dunkers, and the Dunkers defeated the Chargers. The Slammers play the Chargers next. Who do you think will win?

Elicit that most people would expect the Slammers to win but that the facts presented don't rule out the possibility of them losing.

DEVELOPMENT:

Ask, "Compare the original paragraph to: 'The Slammers are all taller than the Dunkers and the Dunkers are all taller than the Chargers.' What can we conclude?"

Elicit that this may be written $S > D$, $D > C$ with the conclusion, $S > C$. (Slammers are all taller than Chargers.) This is an example of a transitive property.

Define: TRANSITIVE PROPERTY (OF EQUALITY) states that if $a = b$ and $b = c$, then $a = c$.

Ask, "What would be a transitive property of congruence?"
 (If $\overline{AB} \cong \overline{CD}$, $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.) Other important properties include:

Define: REFLEXIVE PROPERTY (OF EQUALITY) states that $a = a$. Compare this concept to that of a mirror reflecting an image. Ask, "What would be the reflexive property of congruence?" ($\overline{AB} \cong \overline{AB}$)

Define: SYMMETRIC PROPERTY (OF EQUALITY) states that if $a = b$, then $b = a$. "Express a symmetric property of congruence." (If $\overline{AB} \cong \overline{BC}$, then $\overline{BC} \cong \overline{AB}$.)

Define: EQUIVALENCE RELATION is one for which the reflexive, symmetric and transitive properties all apply.

APPLICATIONS:

Is equality an equivalence relation? (Yes, since the three properties apply to $=$.) Is congruence? (yes) Is "less than"? Why? (No, since it is not true that $a < a$. Note: " $<$ " is transitive but not reflexive or symmetric.)

Is "perpendicular" an equivalence relation? Why? (No, $\overline{AB} \perp \overline{AB}$ is false. Are the symmetric and transitive properties true for " $\perp$ "? - symmetric yes, transitive no.)

1. Are the following relations "equivalence relations"? Why?

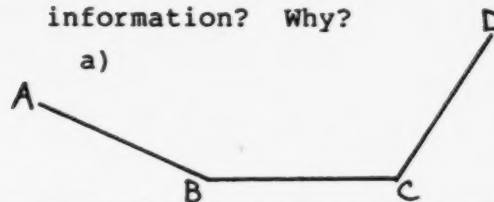
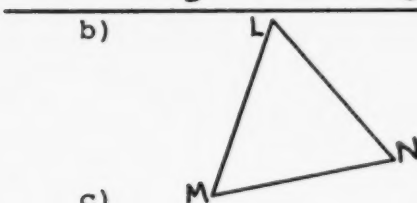
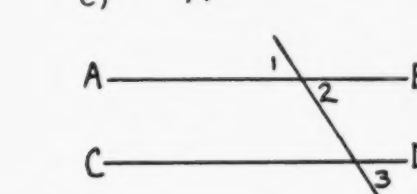
- a) "are the same age" (for people) Ans: yes
- b) "live next door" (for people) Ans: no, not transitive

AL BETH CHIP

Al is next to Beth, Beth is next to Chip, but Al is not next to Chip.

- c) "is a multiple of" (for integers) Ans: no, reflexive and transitive apply, but symmetric does not (4 is a multiple of 2, but 2 is not a multiple of 4)
- d) "is a sister of" (for people) Ans: no, transitive applies, but not reflexive or symmetric (if Maria is a sister of Juan, then Juan is NOT a sister of Maria; he is her brother)

2. What conclusions can be reached using the given information? Why?

a) 	Given that $AB = BC$, $BC = CD$ Ans: $AB = CD$ by the transitive property of equality
b) 	Given that $\overline{LN} \cong \overline{NM}$, $\overline{NM} \cong \overline{LM}$ Ans: $\overline{LN} \cong \overline{LM}$ by the transitive property of congruence
c) 	Given that $\angle 1 \cong \angle 2$, $\angle 2 \cong \angle 3$ Ans: $\angle 1 \cong \angle 3$ by the transitive property of congruence

SUMMARY:

Ask the class to write the three properties of equality discussed today in words.

Reflexive: A quantity is equal to itself.

Symmetric: An equality may be reversed.

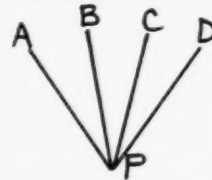
Transitive: If quantities are equal to the same quantity, they are equal to each other.

HOMEWORK:

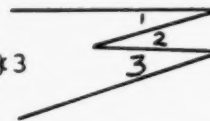
- State the reflexive, symmetric and transitive properties of equality using the variables x , y , z , algebraically.
 $(x = x; (x=y) \rightarrow (y=x); (x=y) \wedge (y=z) \rightarrow (x=z))$
- State in words the reflexive, symmetric and transitive properties of congruence of angles. (an angle $\cong$ to itself; if $\angle A \cong \angle B$ then $\angle B \cong \angle A$; angles $\cong$ to the same angle are $\cong$ to each other)
- Determine if the following are equivalence relations. Explain why or why not.
 - Lives on the same block (for people)
 (yes, all three properties apply)

- b) Is less than (for integers)
(no, not reflexive or symmetric)
- c) Logically equivalent (for statements in logic)
(yes, all three properties apply)
4. Are the reflexive, symmetric and transitive properties for equality and congruence AXIOMS or THEOREMS? Why?
(axioms, since they are accepted without proof)
5. Based on the given information, what conclusion can be reached? Why?

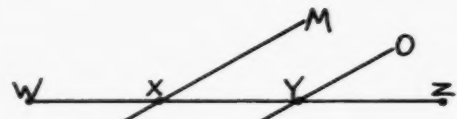
- a) Given: $m\angle APB = m\angle BPC$
and $m\angle BPC = m\angle CPD$
- b) Given: $\overline{CD} \cong \overline{DA}$ and $\overline{DA} \cong \overline{AB}$



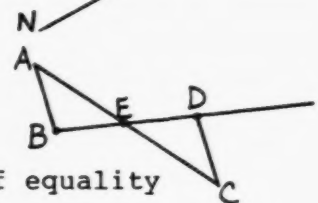
- c) Given: $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$



- d) Given: $WX = XY$ and $XY = YZ$



- e) Given: $\overline{AB} \cong \overline{BE}$, $\overline{BE} \cong \overline{ED}$, $\overline{ED} \cong \overline{DC}$



Conclusions:

- a) $m\angle APB = m\angle CPD$, transitive property of equality
- b) $\overline{CD} \cong \overline{AB}$, transitive property of congruence
- c) $\angle 1 \cong \angle 3$, transitive property of congruence
- d) $WX = YZ$, transitive property of equality
- e) $\overline{AB} \cong \overline{DC}$, transitive property of congruence, applied twice

Lesson 17 - Proofs Involving Lines and Angles

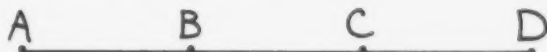
AIM: How do we use the addition and reflexive properties in a simple proof?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Use the addition and reflexive properties to do a simple algebraic proof.
2. Use the addition and reflexive properties to do a simple geometric proof involving line segments.
3. Use the addition and reflexive properties to do a simple geometric proof involving angles.

VOCABULARY: addition property

MOTIVATION:



If we are given that

$\overline{AB} = \overline{CD}$ (with ABCD collinear), what might we conclude about $\overline{AC}$ and $\overline{BD}$? Why?

($\overline{AC} = \overline{BD}$, because we can add $\overline{BC}$ to both $\overline{AB}$ and $\overline{CD}$ and must obtain the same result.)

DEVELOPMENT:

If students don't respond, pose several values for $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$, and ask for $\overline{AC}$ and $\overline{BD}$. (Say $\overline{AB} = 10$, $\overline{BC} = 5$, $\overline{CD} = 10$. Then $\overline{AC} = 15$, $\overline{BD} = 15$, $\overline{AC} \cong \overline{BD}$.)

Ask, "How do we find $\overline{AC}$ and $\overline{BD}$? (by adding: $\overline{AB} + \overline{BC}$ and $\overline{BC} + \overline{CD}$) Have we seen this adding of the same quantity to two values in algebra?" (yes, in equations)

Pose: $x - 10 = 15$ How do we solve?

Elicit:
$$\begin{array}{r} \underline{ + 10} = +10 \\ x = 25 \end{array}$$

Define: **ADDITION PROPERTY:** If the same or equal quantities are added to the same or equal quantities, the results are equal.

In the equation, what quantities were equal? ($x - 10 = 15$)

What same (or equal) quantities were added? ($10 = 10$) What

results are equal? ($x - 10 + 10 = 15 + 10$ or $x = 25$) We can demonstrate this with a formal proof.

<u>Statement</u>	<u>Reason</u>
1. $x - 10 = 15$	1. given equation
2. $10 = 10$	2. reflexive property of equality.
3. $x - 10 + 10 = 15 + 10$ or $x = 25$	3. addition property

How can we use this format to do the geometric proof?

If $\overline{AB} \cong \overline{CD}$, prove $\overline{AC} \cong \overline{BD}$.

<u>Statement</u>	<u>Reason</u>
1. $\overline{AB} \cong \overline{CD}$	1. given
2. $\overline{BC} \cong \overline{BC}$	2. reflexive property of equality.
3. $\overline{AB} + \overline{BC} \cong \overline{BC} + \overline{CD}$ or $\overline{AC} \cong \overline{BD}$	3. addition property

APPLICATIONS:

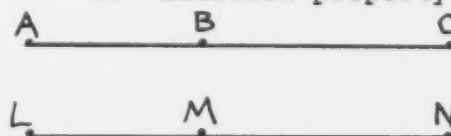
Note, that two angles are $\cong$ if and only if they have the same measures just as for line segments.

1. Given $\angle APB \cong \angle CPD$, what congruence can we deduce about other angles? ($\angle APC \cong \angle BPD$) How is this similar to the example with line segments? (uses "addition")
- Proof:

<u>Statement</u>	<u>Reason</u>
1. $\angle APB \cong \angle CPD$ so $m\angle APB = m\angle CPD$	1. given
2. $m\angle BPC = m\angle BPC$	2. reflexive property of equality.
3. $m\angle APB + m\angle BPC = m\angle CPD + m\angle BPC$ or $m\angle APC = m\angle BPD$ so $\angle APC \cong \angle BPD$	3. addition property



2. Given $\overline{AB} \cong \overline{LM}$, $\overline{BC} \cong \overline{MN}$
Prove $\overline{AC} \cong \overline{LN}$
- Proof:

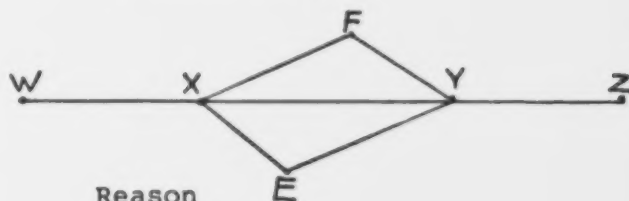


<u>Statement</u>	<u>Reason</u>
1. $\overline{AB} \cong \overline{LM}$	1. given
2. $\overline{BC} \cong \overline{MN}$	
3. $\overline{AB} + \overline{BC} = \overline{LM} + \overline{MN}$ or $\overline{AC} \cong \overline{LN}$	2. addition property

3. Given $\overline{WX} \cong \overline{YZ}$
Prove $\overline{WY} \cong \overline{XZ}$

Proof:

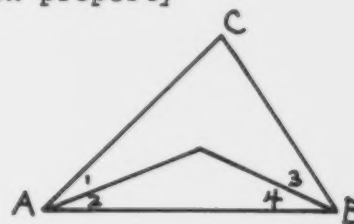
<u>Statement</u>	<u>Reason</u>
1. $\overline{WX} \cong \overline{YZ}$	1. given
2. $\overline{XY} = \overline{XY}$	2. reflexive property of equality.
3. $\overline{WX} + \overline{XY} = \overline{XY} + \overline{YZ}$ or $\overline{WY} \cong \overline{XZ}$	3. addition property



4. Given $m\angle 1 = m\angle 3$, $m\angle 2 = m\angle 4$
Prove $\angle CAB \cong \angle CBA$

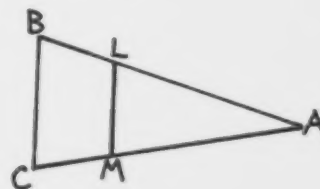
Proof:

<u>Statement</u>	<u>Reason</u>
1. $m\angle 1 = m\angle 3$, $m\angle 2 = m\angle 4$	1. given
2. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$ or $m\angle CAB = m\angle CBA$ so $\angle CAB \cong \angle CBA$	2. addition property



5. Given $\overline{AL} \cong \overline{AM}$, $\overline{BL} \cong \overline{CM}$ for $\triangle ABC$.
What kind of triangle is $\triangle ABC$? (Why?)

Answer: $\triangle ABC$ is isosceles, since $\overline{AB} \cong \overline{AC}$ by the addition property, and a triangle having two congruent legs is isosceles.



6. Prove that if $x - a = b$, then $x = b + a$.

Proof:

<u>Statement</u>	<u>Reason</u>
1. $x - a = b$	1. given
2. $a = a$	2. reflexive property
3. $x - a + a = b + a$ or $x = b + a$	3. addition property

SUMMARY:

Ask the class, "In your own words, what is the addition property? How is it used in proof?"

HOMEWORK:

1. Given: $\overline{AE} \cong \overline{FC}$

Prove: $\overline{AF} \cong \overline{EC}$

2. Given: $\angle AEB \cong \angle DEC$

Prove: $\angle AEC \cong \angle DEB$

3. Given: $\overline{AE} \cong \overline{BE}$, $\overline{EC} \cong \overline{ED}$

Prove: $\overline{AC} \cong \overline{BD}$

4. Given: $\angle DAC \cong \angle CBD$

$\angle CAB \cong \angle DBA$

Prove: $\angle DAB \cong \angle CBA$

5. Given: $\overline{AF} \cong \overline{BF}$, $\overline{FE} \cong \overline{FD}$

Prove: $\overline{AE} \cong \overline{BD}$

6. Given: $\angle CAE \cong \angle CBD$

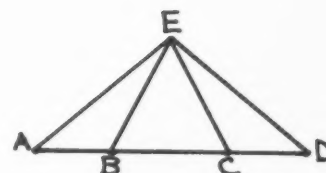
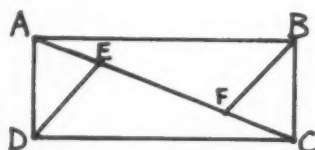
$\angle EAB \cong \angle DBA$

Prove: $\angle CAB \cong \angle CBA$

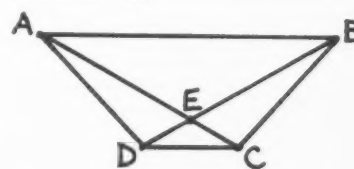
7. Prove that if $ax - b = c$, then $x = \frac{1}{a}(c+b)$

8. Is the addition property an axiom or theorem?

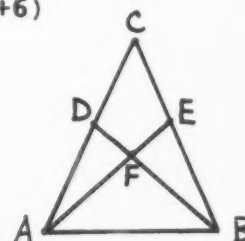
(axiom, since it is accepted without proof)



(figs. for 3+4)



(figs. for 5+6)



Lesson 18 - Proofs Involving Lines and Angles

AIM: How do we use the subtraction property in a simple proof?

PERFORMANCE OBJECTIVES: The student will be able to...

Use the subtraction property to do a simple

- a) algebraic proof
- b) geometric proof involving line segments
- c) geometric proof involving angles

VOCABULARY: subtraction property

MOTIVATION:

If we are given that $\overline{AC} = \overline{BD}$, what can be concluded about $\overline{AB}$ and $\overline{CD}$? Why?



(Note: A, B, C, D are collinear)

Elicit that $\overline{AB} \cong \overline{CD}$ since we can intuitively subtract $\overline{BC}$ from each of $\overline{AC}$ and $\overline{BD}$, with the results $\overline{AB}$ and $\overline{CD}$.

DEVELOPMENT:

Define: **SUBTRACTION PROPERTY:** If the same or equal quantities are subtracted from the same or equal quantities, the results are equal. Which quantities were being subtracted in our example? ($\overline{BC}$, $\overline{BC}$) Why are these congruent? (reflexive property) We set up a formal proof:

Given: $\overline{AC} \cong \overline{BD}$

Prove: $\overline{AB} \cong \overline{CD}$

<u>Statement</u>	<u>Reason</u>
1. $\overline{AC} \cong \overline{BD}$	1. given
2. $\overline{BC} = \overline{BC}$	2. reflexive property of equality.
3. $\overline{AC} - \overline{BC} = \overline{BD} - \overline{BC}$ or $\overline{AB} \cong \overline{CD}$	3. subtraction property

How can we use the subtraction property in an algebraic proof? For example, prove that if $x + a = b$, then $x = b - a$.

- | | |
|----------------|------------------------------------|
| 1. $x + a = b$ | 1. given |
| 2. $a = a$ | 2. reflexive property of equality. |

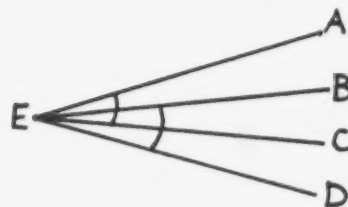
3. $x + a - a = b - a$ 3. subtraction property
or $x = b - a$

APPLICATIONS

1. How could the subtraction property be used with angles?
Elicit a problem like:

Given: $\angle AEC \cong \angle BED$

Prove: $\angle AEB \cong \angle CED$



Note: Encourage students to use various colors in their diagrams.

Proof:

Statement	Reason
1. $\angle AEC \cong \angle BED$ so $m\angle AEC = m\angle BED$	1. given
2. $m\angle BEC = m\angle BEC$	2. reflexive property of equality.
3. $m\angle AEC - m\angle BEC = m\angle BED - m\angle BEC$ or $m\angle AEB - m\angle CED$ so $\angle AEB \cong \angle CED$	3. subtraction property

2. Given: $\overline{AC} \cong \overline{LN}$, $\overline{AB} \cong \overline{MN}$

Prove: $\overline{BC} \cong \overline{LM}$

Proof:

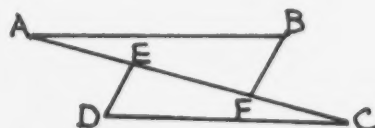


Statement	Reason
1. $\overline{AC} \cong \overline{LN}$, $\overline{AB} \cong \overline{MN}$	1. given
2. $\overline{AC} - \overline{AB} = \overline{LN} - \overline{MN}$ or $\overline{BC} \cong \overline{LM}$	2. subtraction property

3. Given: $\overline{AF} \cong \overline{EC}$

Prove: $\overline{AE} \cong \overline{FC}$

Proof:



Statement	Reason
1. $\overline{AF} \cong \overline{EC}$	1. given
2. $\overline{EF} = \overline{EF}$	2. reflexive property of equality
3. $\overline{AF} - \overline{EF} = \overline{EC} - \overline{EF}$ or $\overline{AE} \cong \overline{FC}$	3. subtraction property

4. Given $\triangle ABC$ is isosceles with legs $\overline{AB}$, $\overline{AC}$; $\overline{AL} \cong \overline{AM}$

Prove: $\overline{LB} \cong \overline{MC}$

Proof:

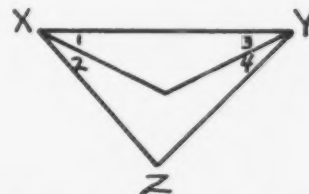


<u>Statement</u>	<u>Reason</u>
1. $\overline{AL} \cong \overline{AM}$; $\triangle ABC$ is isosceles with legs $\overline{AB}$, $\overline{AC}$	1. given
2. $\overline{AB} \cong \overline{AC}$	2. the legs of an isosceles are $\cong$.
3. $\overline{AB} - \overline{AL} = \overline{AC} - \overline{AM}$ or $\overline{LB} \cong \overline{MC}$	3. subtraction property

5. Given: $\angle ZXY = \angle ZYX$,
 $\angle 1 \cong \angle 3$

Prove: $\angle 2 \cong \angle 4$

Proof:



<u>Statement</u>	<u>Reason</u>
1. $\angle ZXY \cong \angle ZYX$, so $m\angle ZXY = m\angle ZYX$ $\angle 1 \cong \angle 3$, so $m\angle 1 = m\angle 3$	1. given
2. $m\angle ZXY - m\angle 1 = m\angle ZYX - m\angle 3$ or $m\angle 2 = m\angle 4$ so $\angle 2 \cong \angle 4$	2. subtraction property

6. Prove that if $x + 2p = 3q$, then $x = 3q - 2p$.

Proof:

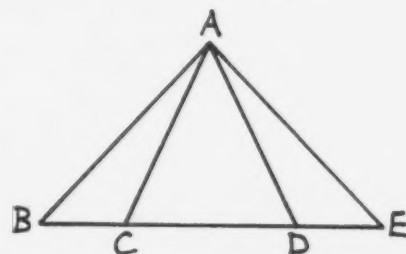
<u>Statement</u>	<u>Reason</u>
1. $x + 2p = 3q$	1. given
2. $2p = 2p$	2. reflexive property of equality.
3. $x + 2p - 2p = 3q - 2p$ or $x = 3q - 2p$	3. subtraction property

7. Given: $\overleftrightarrow{BCDE}$,
 $m\angle BAD = m\angle EAC$, $BC = ED$

Prove: $m\angle BAC = m\angle EAD$,

$BD = EC$

Proof:



<u>Statement</u>	<u>Reason</u>
1. $m\angle BAD = m\angle EAC$	1. given
2. $m\angle CAD = m\angle CAD$	2. reflexive property
3. $m\angle BAD - m\angle CAD =$ $m\angle EAC - m\angle CAD$ or $m\angle BAC = m\angle EAD$	3. subtraction property
4. $\overleftrightarrow{BCDE}$, $BC = ED$	4. given

5. $CD = CD$

5. reflexive property of equality.

6. $BC + CD = ED + CD$ or
 $BD = EC$

6. addition property

Note: This proof is presented in "equal" terminology rather than "congruent" terminology to provide comparison and contrast of notation. The teacher may wish to have the class study the same proof in "congruence" notation as well.

SUMMARY:

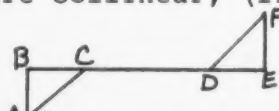
Ask: What does the subtraction property say? How does it help us to conclude that line segments or angles are congruent?

HOMEWORK:

1. Given: B, C, D, E are collinear, (fig. for EX 1 2)

$\overline{BD} \cong \overline{CE}$

Prove: $\overline{BC} \cong \overline{DE}$



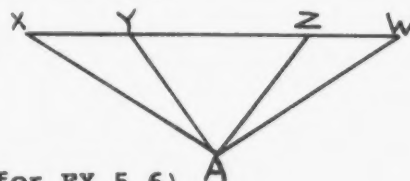
2. Given: $\angle BCD \cong \angle EDC$, $\angle BCA \cong \angle EDF$

Prove: $\angle ACD \cong \angle FDC$

3. Given: $\overline{XZ} \cong \overline{YW}$

Prove: $\overline{XY} \cong \overline{ZW}$

(fig. for EX 3 4)

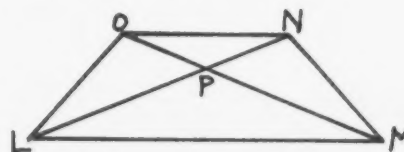


4. Given: $\angle XAZ \cong \angle WAY$

Prove: $\angle XAY \cong \angle WAZ$

5. Given: $\overline{LN} \cong \overline{MO}$, $\overline{NP} \cong \overline{OP}$ (fig. for EX 5 6)

Prove: $\overline{LP} \cong \overline{MP}$



6. Given: $\angle OLM \cong \angle NML$,

$\angle NLM \cong \angle OML$

Prove: $\angle NLO \cong \angle OMN$

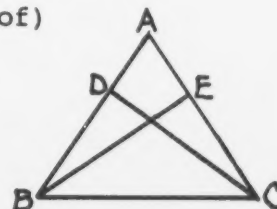
7. Prove that if $ax + b = c$, then $x = \frac{1}{a}(c+b)$.

8. Is the subtraction property an axiom or a theorem?
(axiom, since it is accepted without proof)

9. Given: $m\angle ABE = m\angle ACD$,

$m\angle EBC = m\angle DCB$, $AB = AC$, $DB = EC$

Prove: $m\angle ABC = m\angle ACB$, $AD = AE$



Lesson 19 - Proofs Involving Lines and Angles

AIM: How do we prove that angles are congruent?

PERFORMANCE OBJECTIVES: The student will be able to...

Prove the following theorems (algebraically) and use them in other proofs.

- All right angles are congruent.
- If 2 angles are vertical angles, then they are congruent.
- If 2 angles are complementary to the same (or congruent) angle(s), they are congruent.
- If 2 angles are supplementary to the same (or congruent) angle(s), they are congruent.

VOCABULARY: no new terms

MOTIVATION:

Sue bisected a straight angle with a ruler and compass. What angles resulted from this construction? What is true about these angles? (two right angles which are congruent to each other)

DEVELOPMENT:

Ask the class to describe the circumstances under which angles are congruent (if their measures are equal). Why then are all right angles congruent? (because their measures are equal)

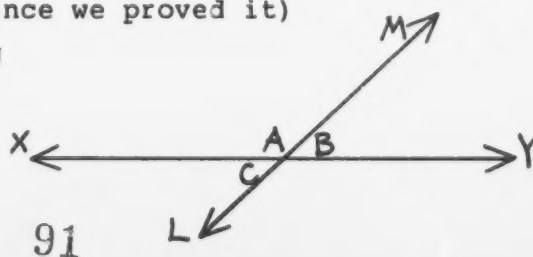
Proof: Given $\angle A$, $\angle B$ are right angles.

Prove $\angle A = \angle B$.

We may use an algebraic approach. $m\angle A = 90$, $m\angle B = 90$ since the measure of a right angle is 90. Therefore, $m\angle A = m\angle B$ by the transitive property. Therefore, $\angle A \cong \angle B$ by the definition of congruent angles. Is the original statement a theorem or axiom? (theorem, since we proved it)

Consider two lines intersecting to form vertical angles. If

$m\angle A = 110$, find $m\angle B$.



Since the non-common sides of $\angle A$ and $\angle B$ form opposite rays and $\angle A$ and $\angle B$ are adjacent, $\angle A$ and $\angle B$ are supplementary.

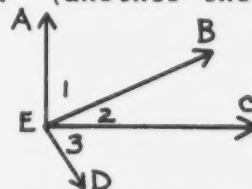
Therefore, $m\angle B = 180 - m\angle A = 180 - 110 = 70$.

By similar reasoning, $m\angle C = 110$. What relationship exists between $\angle A$ and $\angle C$? (congruent) We can generalize and prove this:

<u>Statement</u>	<u>Reason</u>
1. $m\angle A + m\angle B = 180$ $m\angle B + m\angle C = 180$	1. since XY, LM are straight lines
2. $m\angle A + m\angle B =$ $m\angle B + m\angle C$	2. by the transitive property of equality.
3. $m\angle B = m\angle B$	3. why? (reflexive property) of equality.
4. $m\angle A = m\angle C$	4. why? (subtraction property)
5. $\angle A \cong \angle C$	5. since the measures are equal

As a result, we have proved that: If two angles are vertical angles, then they are congruent. (another theorem)

Let's consider another kind of angle pair. What must be true about $\angle 1$ and $\angle 3$ if they are both complementary to $\angle 2$?



Elicit $\angle 1 \cong \angle 3$. For example, if $m\angle 1 = 70$, then $m\angle 2 = 20$, since they are complementary. But since $\angle 2$ and $\angle 3$ are complementary, then $m\angle 3 = 70$, which is the same as $m\angle 1$.

In general: $m\angle 1 + m\angle 2 = 90$ and $m\angle 2 + m\angle 3 = 90$

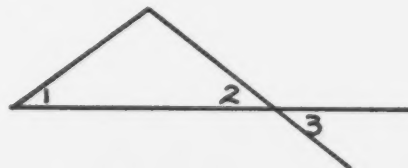
Therefore: $m\angle 1 = 90 - m\angle 2$ and $m\angle 3 = 90 - m\angle 2$

So: $m\angle 1 = m\angle 3$

This proves the theorem that if two angles are complementary to the same (or congruent) angles, they are congruent. If we replace "complementary" with "supplementary", we obtain another theorem. (Prove it in homework.)

APPLICATIONS:

1. Given: $\angle 1 \cong \angle 2$
Prove: $\angle 1 \cong \angle 3$



Proof:

<u>Statement</u>	<u>Reason</u>
1) $\angle 1 \cong \angle 2$	1) given
2) $\angle 2 \cong \angle 3$	2) if 2 angles are vertical, they are congruent
3) $\angle 1 \cong \angle 3$	3) transitive property of congruence.

2. Given: $\angle 1 \cong \angle 2$, $\angle 4 \cong \angle 3$

Prove: $\angle 1 \cong \angle 4$

Fill in missing reasons:

Proof:

<u>Statement</u>	<u>Reason</u>
1) $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$	1) given
2) $\angle 2 \cong \angle 3$	2) if angles are vertical angles, they are $\cong$
3) $\angle 1 \cong \angle 3$	3) transitive property of congruence.
4) $\angle 1 \cong \angle 4$	4) transitive property of congruence.

Note: It is possible to omit step 3 and go directly from step 2 to step 4.

3. If $\angle DAB$ is a right angle, what relationship exists between $\angle 1$ and $\angle 2$? (complementary)

Given: $\angle DAB$, $\angle CBA$ are right angles, $\angle 1 = \angle 4$.

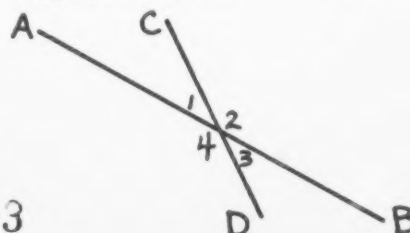
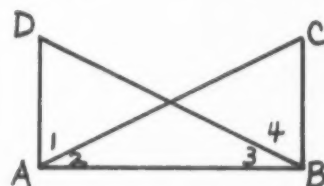
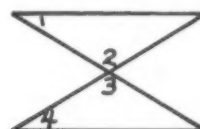
Prove: $\angle 2 \cong \angle 3$.

Proof:

<u>Statement</u>	<u>Reason</u>
1) $\angle DAB$, $\angle CBA$ rt. $\angle$'s $\angle 1 \cong \angle 4$	1) given
2) $\angle 1$, $\angle 2$ are complementary and $\angle 3$, $\angle 4$ are complementary	2) two angles whose measures total 90 (a right angle) are complementary.
3) $\angle 2 \cong \angle 3$	3) angles complementary to $\cong$ angles are $\cong$.

4. Given: $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$

a) If $m\angle 1 = 30$, find $m\angle 2$,
 $m\angle 3$, $m\angle 4$ (150, 30, 150)



- b) If $m\angle 2 = 160$, find $m\angle 1$,
 $m\angle 3$, $m\angle 4$ (20, 20, 160)
- c) If $m\angle 1 = x + 10$ and $m\angle 3 = 4x - 20$, find x , find $m\angle 1$
 $(x + 10 = 4x - 20, x = 10; x + 10 = 20)$
- d) If $m\angle 3 = 5x - 25$ and $m\angle 1 = 7x - 65$, solve for x ,
 find $m\angle 3$, $m\angle 4$
 $(5x - 25 = 7x - 65, x = 20; m\angle 3 = 75, m\angle 4 = 105)$

SUMMARY:

Have students list the four theorems.

- All right angles are congruent.
- If 2 angles are vertical angles, then they are congruent.
- If 2 angles are complementary to the same or congruent angles, they are congruent.
- If two angles are supplementary to the same or congruent angles, they are congruent.

HOMEWORK:

Complete the sentences.

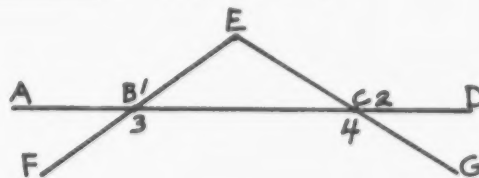
- If angles are vertical angles, they are _____.
 (congruent)
- If 2 angles are supplements of the same angle, the 2 angles are _____. (congruent)
- All right angles are _____. (congruent)
- Prove #2 algebraically.

5. Given: $\angle 1 \cong \angle 2$

Prove: $\angle 3 \cong \angle 4$

6. Given: $\angle 1 \cong \angle 2$, $\overleftrightarrow{AB}$

Prove: $\angle 3 \cong \angle 4$



7. In the diagram for #5, find the measures of $\angle EBC$, $\angle CBF$ and $\angle FBA$ if $m\angle EBA = 115$. (65, 115, 65)

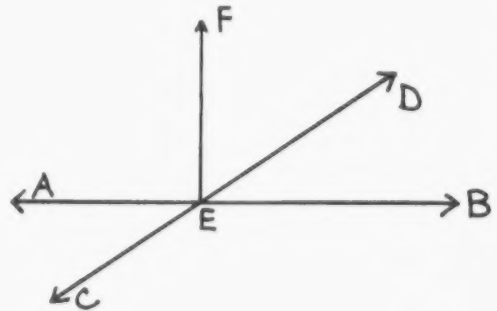
8. In the diagram for #5, find the measure of $\angle EBC$ and $\angle CBF$ if $m\angle EBC = 2x + 20$ and $m\angle ABF = 5x - 40$.

$$(5x - 40 = 2x + 20, x = 20; m\angle EBC = 60, m\angle CBF = 120)$$

9. Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$
intersect at E, $EF \perp AB$,
 $m\angle DEB = 2x + 15$,
 $m\angle BEC = 7x + 75$

Find each of the following:

- $m\angle DEB$ (35)
- $m\angle BEC$ (145)
- $m\angle CEA$ (35)
- $m\angle AEF$ (90)
- $m\angle AED$ (145)
- $m\angle FED$ (55)
- Name a pair of vertical angles. ($\angle DEB$, $\angle AEC$)
- Name a pair of complementary angles. ($\angle FED$, $\angle DEB$)
- Name a pair of supplementary angles ($\angle DEB$, $\angle BEC$)



Lesson 20 - Perpendicular Lines

AIM: How do we do simple proofs involving perpendicular lines and altitudes?

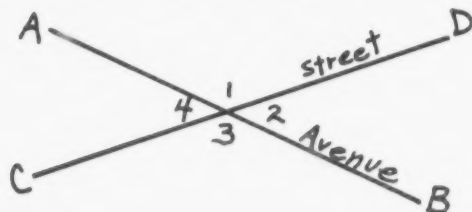
PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove angles are congruent when it is given that two lines are perpendicular.
2. Prove angles are congruent when it is given that a segment is an altitude of a triangle.

VOCABULARY: perpendicular lines, altitude

MOTIVATION:

If two streets intersect it is best for engineers to arrange them so that the view of oncoming traffic will be at a maximum. What should be true of the angles formed in order for a driver to have the best view?

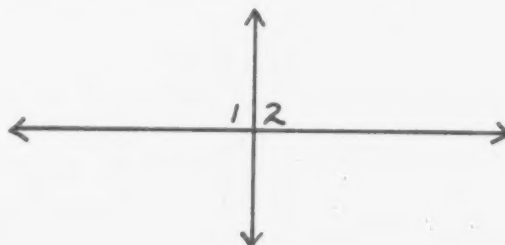


DEVELOPMENT:

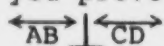
1. Continue the motivation problem. Elicit (in any order):
 - a) All the angles are right angles.
 - b) All the angles are congruent.
 - c) The lines are perpendicular.

Ask, "Could the angles be congruent but the lines not be perpendicular? How do we know?" Have students outline a proof:

$$\begin{aligned}\angle 1 &\cong \angle 2 \\ m\angle 1 &= m\angle 2 \\ m\angle 1 + m\angle 2 &= 180 \\ 2(m\angle 1) &= 180 \\ m\angle 1 &= 90 \\ \therefore \text{lines are } \perp\end{aligned}$$



Ask, "If we knew the lines in the diagram are perpendicular, would the angles have to be congruent? Could you prove it?"



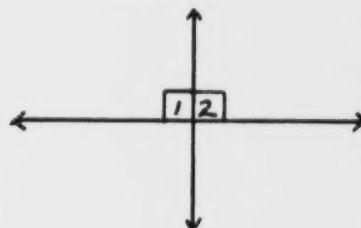
$\angle 1$ and $\angle 2$ are right angles.

$$m\angle 1 = 90$$

$$m\angle 2 = 90$$

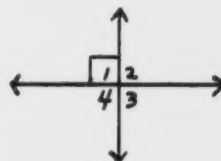
$$m\angle 1 = m\angle 2$$

$$\angle 1 \cong \angle 2$$



2. Have the class consider:

If two intersecting lines form one right angle, do they form any others?



3. Have students state the above theorems. (Check the phrasing in your textbook. The treatments vary quite a bit from text to text.) Possible statements include:

If two lines are perpendicular, then they form congruent adjacent angles.

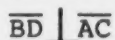
Converse of above.

Two perpendicular lines form four right angles.

4. Do Application #1.

5. $\triangle ABC$ represents a pyramid. Which line segment represents the height of the pyramid? What name do we give to that line segment? (altitude) What is true of $\angle 1$ and $\angle 2$? Could you prove it?

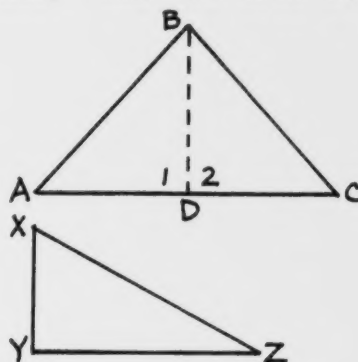
$\overline{BD}$ is an altitude.



$\angle 1$ and $\angle 2$ are right angles.

$$\angle 1 \cong \angle 2$$

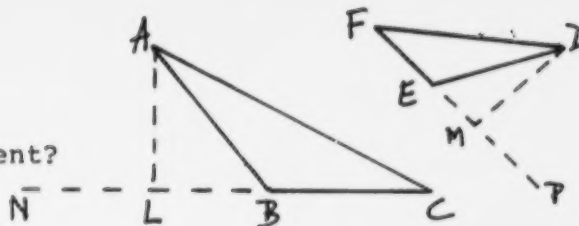
If $\triangle XYZ$ is a pyramid in a drawing turned sideways, draw the altitude from Y to $\overline{XZ}$.



Practice drawing other altitudes, including the case in which the altitude falls outside the triangle.

Which angles are congruent?

6. Do Application #2.



7. By the way, how many perpendicular lines can you draw from a point to a line?

Elicit the postulate: From an exterior point there is one and only one line perpendicular to a given line.

8. Do Application #3.

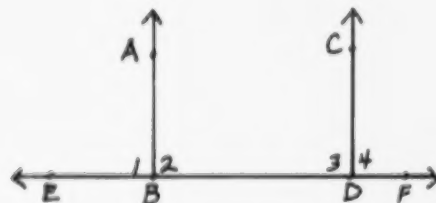
APPLICATIONS

1. Given: $\overline{AB} \perp \overline{EF}$

$\overline{CD} \perp \overline{EF}$

Prove: $\angle 1 \cong \angle 4$

($\perp$ lines form right $\angle$ s and all right $\angle$ s are $\cong$)

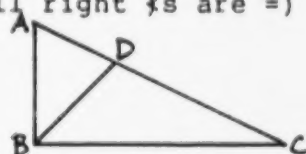


2. Given: $\overline{AB} \perp \overline{BC}$

$\overline{BD}$ is an altitude.

Prove: $\angle ABC \cong \angle ADB$

($\perp$ lines form right $\angle$ s, altitude forms right $\angle$, all right $\angle$ s are $\cong$)

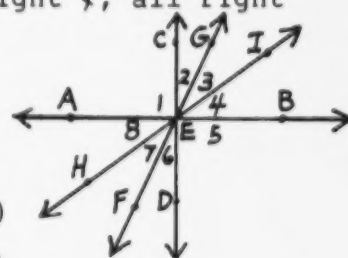


3. Given: $\overline{AB} \perp \overline{CD}$, $\overline{FG} \perp \overline{HI}$

$\overline{AB} \perp \overline{CD}$

Prove: $m\angle 4 + m\angle 7 + m\angle 2 = m\angle 1$

($\perp$ lines form right $\angle$, vertical $\angle$ s are $\cong$)



4. Given: $\overline{AB} \perp \overline{BC}$

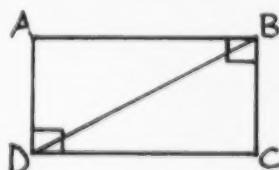
$\overline{AD} \perp \overline{DC}$

$\angle ABD \cong \angle ADB$

Prove: $\angle CBD \cong \angle CDB$

Do $\angle A$ and $\angle C$ have to be right angles? (no) Can you draw a figure where $\angle A$ and $\angle C$ are not right angles?

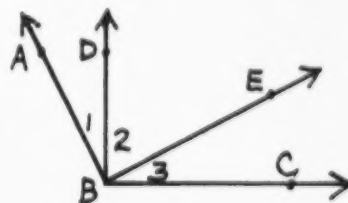
($\perp$ lines form right $\angle$ s, subtraction property)



5. Given: $\overline{DB} \perp \overline{BC}$

$\overline{EB} \perp \overline{AB}$

Prove: $m\angle 1 = m\angle 3$



($\perp$ lines form right $\angle$ s, reflexive property, subtraction property)

6. How would you use a plumb line to find out if a beam in a house is horizontal? (Measure $\angle$ s formed by plumb line and beam; if the beam is horizontal the $\angle$ s will be right $\angle$ s.)

7. In the room, name perpendicular lines.

Is $\overline{AB} \perp \overline{GF}$?

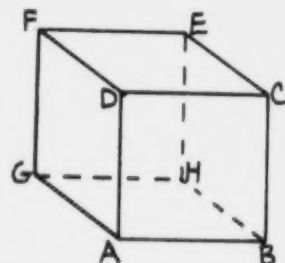
(No, they are skew lines.)

8. State a problem without the diagram:

Given: $\overline{AB} \perp \overline{CD}$ at E.

Prove: $m\angle AED + m\angle CEB = 180$

($\perp$ lines form right $\angle$ s, whose measure is 90, and $90 + 90 = 180$)

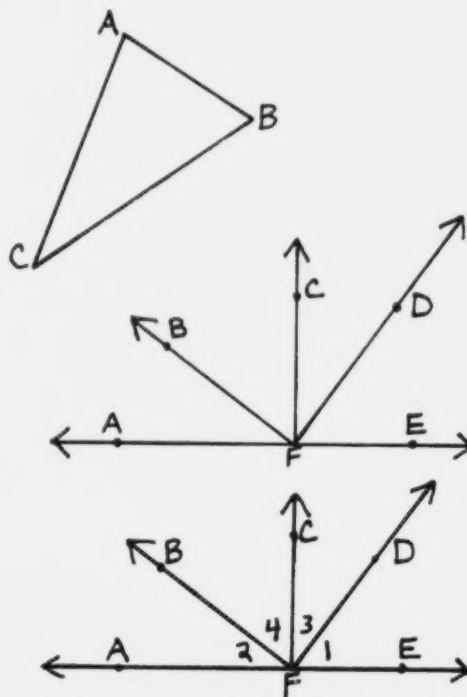


SUMMARY:

- How many altitudes does a scalene triangle have? If all three are drawn, how many right angles are formed? (6)
- Depending on the ability level of the class, choose one of the applications to summarize and restate the theorems developed.

HOMEWORK:

- Construct the altitude from A and label it $\overline{AD}$.
 - Name the congruent angles.
- Given: $\overline{AF} \perp \overline{CF}$, $\angle AFE$
 $m\angle CFD = 40$
 $m\angle BFA = 40$
 Find $m\angle BFC$ and $m\angle DFE$
- Given: $\overline{AF} \perp \overline{CF}$, $\angle AFE$
 $\angle 1 \cong \angle 4$
 Prove: $\angle 3 \cong \angle 2$



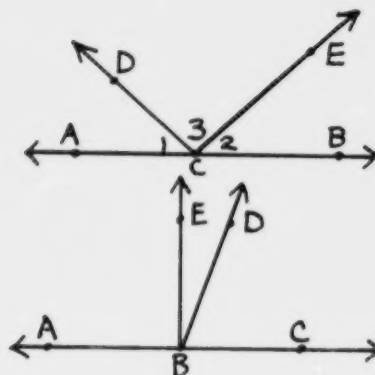
4. Given: $\overline{DC} \perp \overline{CE}$, $\overleftrightarrow{ACB}$

Prove: $\angle 1$ and $\angle 2$ are complementary.

5. Given: $\overleftrightarrow{ABC}$

$\angle EBD$ and $\angle DBC$ are complementary

Prove: $\overline{EB} \perp \overline{AC}$



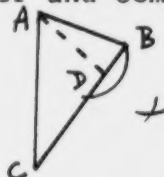
6. Given: In acute triangle ABC, altitude $\overline{AD}$ is drawn. $\overline{DE}$ is drawn $\perp \overline{AB}$.

Prove: $\angle ADC \cong \angle DEA$

HOMEWORK SOLUTIONS:

1. With a ruler and compass:

a)



b) $\angle ADC \cong \angle ADB$

2. $m\angle BFC = 50 = m\angle DFE$

3. $\overline{AF} \perp \overline{CF}$, $\overleftrightarrow{AE}$

$\angle AFC$ and $\angle EFC$ are right angles.

$\angle 1$ compl $\angle 3$

$\angle 2$ compl $\angle 4$

$\angle 1 \cong \angle 4$

$\therefore \angle 3 \cong \angle 2$

4. $\overleftrightarrow{ACB}$

$\angle ACB$ is a straight angle.

$m\angle 1 + m\angle 3 + m\angle 2 = m\angle ACB$

$m\angle ACB = 180$

$\overline{DC} \perp \overline{CE}$

$\angle 3$ is a right angle.

$m\angle 3 = 90$

$m\angle 1 + m\angle 2 = 90$

$\angle 1$ and $\angle 2$ are compl

5. $\angle EBD$ and $\angle DBC$ are complementary

$$m\angle EBD + m\angle DBC = 90$$

$$m\angle EBD + m\angle DBC = m\angle EBC$$

$$m\angle EBC = 90$$

$$\therefore \overline{EB} \perp \overline{AC}$$

6. $\overline{AD}$ is an altitude

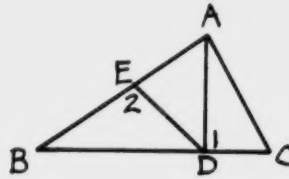
$$\overline{AD} \perp \overline{CB}$$

$\angle 1$ is a right angle

$$\overline{DE} \perp \overline{AB}$$

$\angle 2$ is a right angle

$$\angle 1 \cong \angle 2$$



Lesson 21 - Perpendicular Lines

AIM: How do we do simple proofs involving bisectors and midpoints?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Do a simple proof in algebra using "Halves of equal quantities are equal" (Multiplicative Property).
2. Do a simple geometric proof when the following is given:
 - a) A midpoint of a line segment.
 - b) A bisector of a line segment.
 - c) A perpendicular bisector of a line segment.
 - d) An angle bisector.

VOCABULARY: perpendicular bisector, counter example

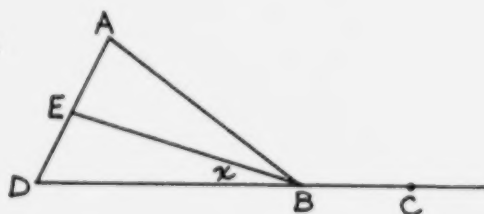
MOTIVATION:

$\overline{BE}$ bisects $\angle DBA$

$m\angle ABC = 154$

$m\angle EBD = x$

Find x .



DEVELOPMENT:

1. Solve the motivation problem. Justify each step.

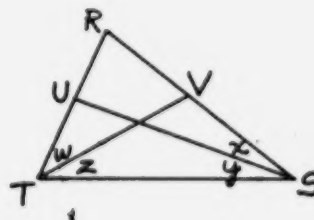
$$2x + 154 = 180$$

$$2x = 26$$

$$x = 13$$

Elicit in step 3 that, "Halves of equal quantities are equal. (Multiplicative Property)" Must $\overline{BE}$ bisect $\overline{AD}$? (No, see Homework problem #5.) Could $\overline{BE}$ bisect $\overline{AD}$? (Yes.) When? (If $\overline{AB} \cong \overline{BD}$.)

2. In $\triangle RST$, two angle bisectors are drawn. Which angles are congruent? ($\angle w \cong \angle z$, $\angle x \cong \angle y$)



If $\angle RTS$ and $\angle RST$ are congruent, which other angles are congruent? ($\angle w \cong \angle x \cong \angle y \cong \angle z$) Could you prove it? Write the steps of the proof.

3. State the definition: An angle bisector of a triangle is a line segment starting at a vertex which bisects an angle and terminates in the opposite side.

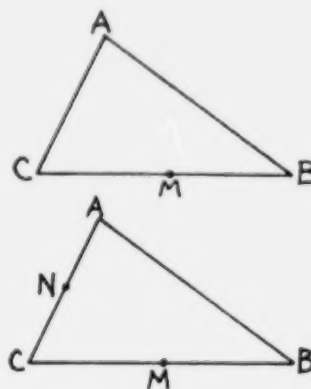
4. Do Application #1.

5. If M is a midpoint in $\triangle ABC$, which quantities are congruent? ($\overline{CM} \cong \overline{BM}$) Continue the problem.

If M and N are midpoints and $\overline{AC} \cong \overline{BC}$,

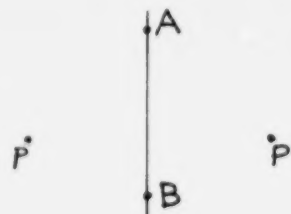
Prove: $\overline{BM} \cong \overline{AN}$

(Multiplicative Property)



6. Do Application #2.

7. Suppose you are standing in front of a mirror which is represented by a line, $\overleftrightarrow{AB}$, and you are represented by a point, P.



Where does your image appear to be? (P') What is the relation between the lines? Elicit that $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{PP'}$.

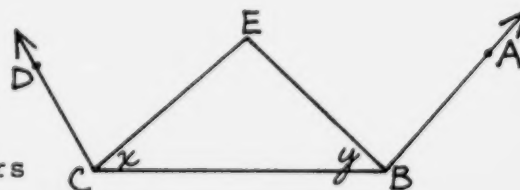
8. Do Application #3.

9. Have the class demonstrate that a median in a triangle is not necessarily an angle bisector.

10. Be sure your students understand the difference between a midpoint and a bisector. A bisector of a line segment is the line or line segment that intersects the segment at its midpoint.

APPLICATIONS:

1. Given: $\angle ABC \cong \angle DCB$
 $\overline{EB}$ and $\overline{EC}$ are angle bisectors

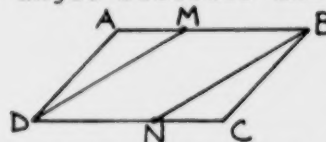


Prove: $\angle x \cong \angle y$ (definition of angle bisector and halves of equals are equal)

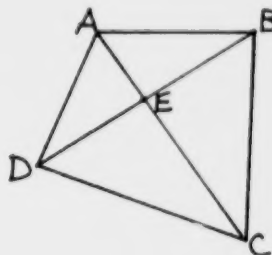
2. Given: $\overline{AB} \cong \overline{CD}$

M and N are midpoints

Prove: $\overline{MB} \cong \overline{ND}$ (definition of midpoint and halves of equals are equal)



3. In the kite at the right, $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$. State which two quantities are congruent and prove it. ($\overline{BE} \cong \overline{DE}$, definition of bisector)

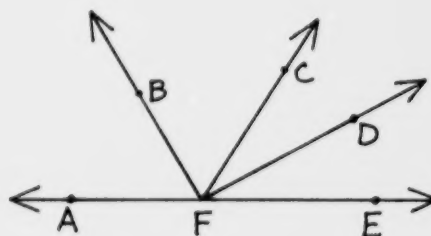


4. Given: $\overleftrightarrow{AFE}$

$\overline{FB}$ bisects $\angle AFC$

$\overline{FD}$ bisects $\angle CFE$

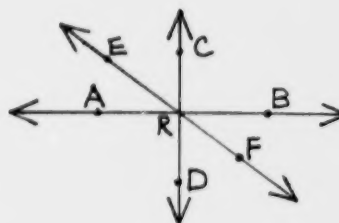
What is true about $\overline{FB}$ and $\overline{FD}$? Prove it. ($\overline{FB} \perp \overline{FD}$, half of a straight $\angle$ is a right $\angle$)



5. Given: $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$

$\overleftrightarrow{ERF}$ bisects $\angle ARC$

Find: $m\angle DRF$ (45, vertical $\angle$ s are $\cong$, half of 90 is 45)
Prove your answer.



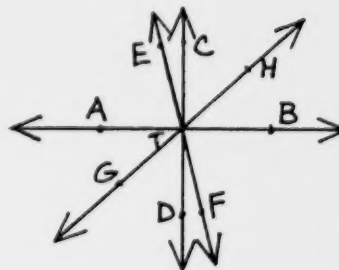
6. Given: $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$, $\overleftrightarrow{ETF}$

$\overleftrightarrow{HTG}$ bisects $\angle ATD$

$m\angle ETC = 20$

Find: measures of all the angles

($m\angle ATE = m\angle BTF = 70$, $m\angle ETC = m\angle FTD = 20$, $m\angle CTH = m\angle DTG = m\angle HTB = m\angle GTA = 45$)



SUMMARY:

$\overleftrightarrow{BE}$ is the perpendicular bisector of $\overline{AD}$
 $\overleftrightarrow{EC}$ bisects $\angle BED$

1. Name the congruent line segments.

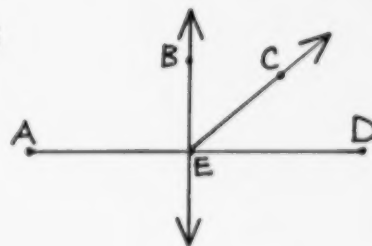
$$(\overline{AE} \cong \overline{DE})$$

2. Name pairs of congruent angles.

$$(\angle AEB \cong \angle DEB, \angle BEC \cong \angle DEC)$$

Justify your answers.

($\perp$ lines form right angles; bisection definitions)



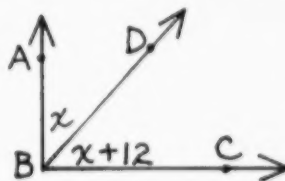
HOMEWORK:

1. Given: $\overline{AB} \perp \overline{BC}$

$$m\angle ABD = x$$

$$m\angle DBC = x + 12$$

$$\text{Prove: } x = 39$$

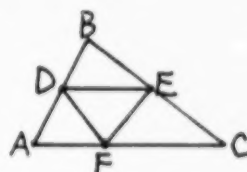


2. Given: E is the midpoint of $\overline{BC}$.

F is the midpoint of $\overline{AC}$.

$$\overline{BC} \cong \overline{AC}$$

$$\text{Prove: } \overline{BE} \cong \overline{AF}$$



3. a) Given: $\overleftrightarrow{EB}$ bisects $\angle AEC$

$\overleftrightarrow{EC}$ bisects $\angle BED$

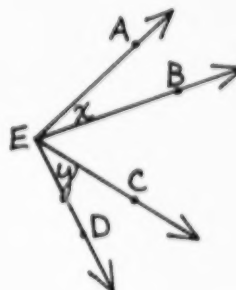
$$m\angle AEB = 40$$

$$\text{Find: } m\angle DEC$$

- b) Given: $\overleftrightarrow{EB}$ bisects $\angle AEC$

$\overleftrightarrow{EC}$ bisects $\angle BED$

$$\text{Prove: } \angle x = \angle y$$



4. Given: $\overleftrightarrow{AO}$ bisects $\angle BAC$,

$$m\angle BAO = 6x + 5 \text{ and } m\angle CAO = 3x + 20$$

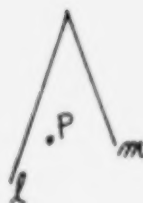
$$\text{Find: } m\angle BAO$$

5. One student made the incorrect assumption that an angle bisector in a triangle bisects the opposite side. How would you show the assumption is false through construction?

6. Find the reflection of A in line ℓ .



7. Find the reflection, P' , of P in line ℓ and the reflection of P' in line m .



HOMEWORK SOLUTIONS:

1. $\overline{AB} \perp \overline{BC}$

Given

$\angle ABC$ is a right $\angle$

Perpendicular lines form right angles.

$$x + x + 12 = 90$$

Partition Postulate

$$2x + 12 = 90$$

Distributive Law

$$2x = 78$$

Subtraction Axiom

$$x = 39$$

Halves of equals are equal.

2. E and F are midpoints

Given

$$BE = EC$$

Given

$$BE = \frac{1}{2}BC$$

A midpoint divides a

$$AF = \frac{1}{2}AC$$

line segment in half.

$$BE = AF$$

Halves of equals are equal.

3. a) $m\angle x = 40 = m\angle BEC = m\angle y$

- b) $\overline{EB}$ bisects $\angle AEC$

Given

$\overline{EC}$ bisects $\angle BED$

Given

$$\angle x \cong \angle BEC$$

A bisector divides an angle into two congruent angles.

$$\angle y \cong \angle BEC$$

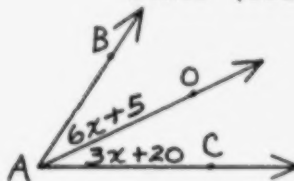
Two angles congruent to the same angle are congruent to each other (Transitivity).

$$\angle x \cong \angle y$$

4. $6x + 5 = 3x + 20$

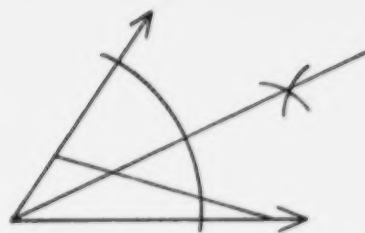
$$3x = 15$$

$$x = 5$$

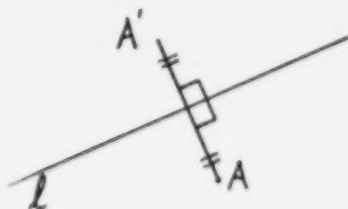


$$m\angle BAO = 6X + 5 = 35$$

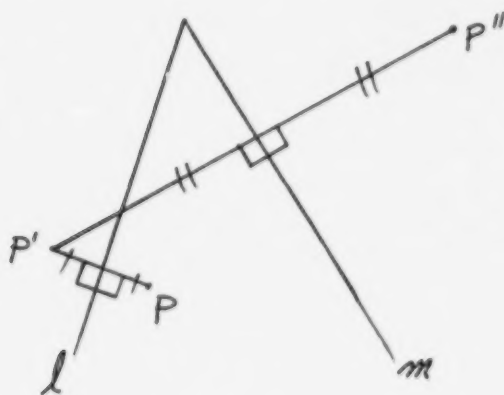
5. Construct an angle bisector in a scalene triangle. The opposite side will not be bisected.



6.



7.



Lesson 22 - The Formal Proof

AIM: How do we write a formal proof in geometry?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Write a formal geometric proof using theorems and definitions concerning line segments and angles.
2. Apply the concepts from recent lessons to slightly more difficult problems.

VOCABULARY: formal proof, counter example, hypothesis, conclusion

MOTIVATION: Which of these statements is true?

1. The formula, $P = n^2 + n + 17$, will always produce a prime number if n is a natural number.
2. There are an infinite number of prime numbers.

DEVELOPMENT:

1. Discuss the motivation problems: How do we know if they are true? Try a few natural numbers in the formula. Discuss induction and its tentativeness. Ask, "Can anyone think of a number that would make the formula false? Does the formula suggest any special number?"
(17) If $n = 17$, then $P = 17^2 + 17 + 17$ is divisible by 17. Say, "We call such an example that disproves a statement a 'counter-example'."
Ask, "Do you believe the second statement? (Most students will indicate that it is true.) How can we be sure it is true?" Discuss the necessity of proving the statement. A brief indirect proof might be given: Assume that there are only a finite number, n , of primes, $\{P = P_1, P_2 \dots P_n\}$. Create a new number which is a prime but not in the set, $P_1 \cdot P_2 \cdot P_3 \dots P_n + 1$, or one of its divisors > 1 .
(The expression is either a prime or has a divisor, if

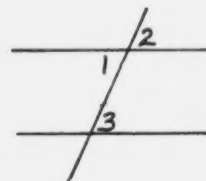
composite, that is greater than P_n . $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 + 1 = 510511 = 19 \times 97 \times 277$ which is a contradiction.)

2. Compare a formal proof in mathematics to a case in a court of law. The analogy is useful in developing the spirit and technique of a proof. If time permits, you might actually show a parallel development of a legal case and a geometric proof:

Given: Police tell you
there was an accident on
Thursday. Two cars
collided at an intersection
Your client was seen
leaving the scene of the
accident.

Prove: Your client is
innocent.

Given: $\angle 1 = \angle 3$



Prove: $\angle 2 \cong \angle 3$

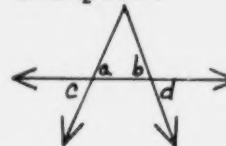
- 1) There was an accident on Thursday.
- 2) Your client was in Chicago on Thursday.
- 3) Your client is innocent.

- 1) Given by police.
- 2) Witnesses will testify.
- 3) Assumption: He could not have been in Chicago and committed crime.

3. Illustrate the formal proof with examples.

a) Given: $\angle a \cong \angle b$

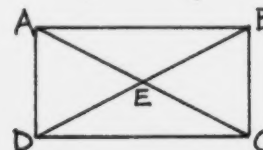
Prove: $\angle c \cong \angle d$



b) Given: $\overline{AC} \cong \overline{BD}$

$\overline{AC}$ and $\overline{BD}$ bisect each other.

Prove: $\overline{AE} \cong \overline{ED}$



Describe the parts of a formal proof:

Diagram

Given (hypothesis)

Prove (conclusion)

Plan or Analysis of Proof

Statements

Reasons

Discuss the need for a plan. Suggest an analytical approach in which you work backwards from the conclusion.

4. Ask, "What kinds of reasons can we use in a proof?"
(hypotheses, definitions, axioms, postulates and theorems)

5. Do Application #1.

Show how a plan can be developed by thinking of the conclusion first: $\overline{AD}$ will be perpendicular to $\overline{EC}$ only if $\angle x$ and $\angle y$ are right angles. They will be right angles if they are congruent. How can we show that they are congruent?

APPLICATIONS:

1. Given: $\angle x \cong \angle z$

$$\angle y \cong \angle z$$

Prove: $\overline{AD} \perp \overline{EC}$

2. Given: $\overline{DB} \perp \overline{BC}$

$$\overline{EC} \perp \overline{BC}$$

$$\angle w \cong \angle z$$

Prove: $\angle x \cong \angle y$

3. Given:

C is the midpoint of $\overline{BD}$.

D is the midpoint of $\overline{EC}$.

Prove: $\overline{ED} \cong \overline{BC}$

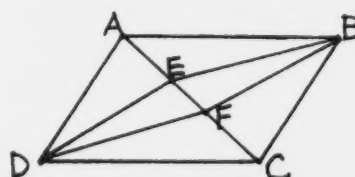
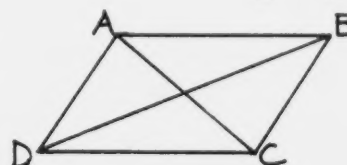
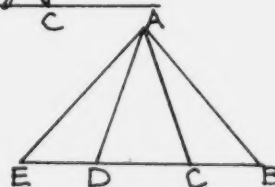
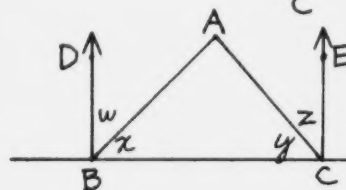
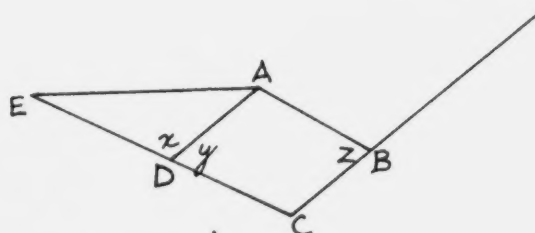
4. Given: $\angle BAC \cong \angle DCA$

$$\angle DAC \cong \angle BCA$$

Prove: $\angle DAB \cong \angle BCD$

5. Given: $\overline{AE} \cong \overline{CF}$

Prove: $\overline{AF} \cong \overline{CE}$



SUMMARY:

1. Why do we need to prove things in mathematics?
2. What are the parts of a formal proof?

HOMEWORK:

1. Show that $P = n^2 - n + 41$ does not always give a prime number for all natural numbers n .

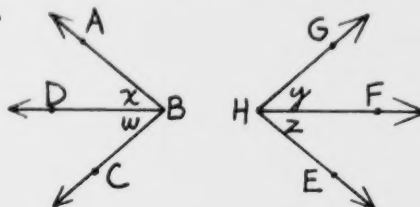
2. Supply the reasons in the proof.

Given: $\overline{AB} \perp \overline{BC}$

$\overline{GH} \perp \overline{HE}$

$\overline{BD}$ and $\overline{HF}$ are angle bisectors.

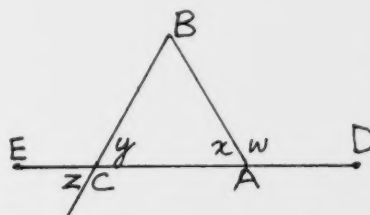
Prove: $\angle x \cong \angle y$



- a) $\overline{AB} \perp \overline{BC}$
 $\overline{GH} \perp \overline{HE}$
- b) $\angle ABC$ is a right $\angle$
 $\angle GHE$ is a right $\angle$
- c) $\angle ABC \cong \angle GHE$
- d) $\overline{BD}$ and $\overline{HF}$ are $\angle$ bisectors
- e) $m\angle x = \frac{1}{2}m\angle ABC$
 $m\angle y = \frac{1}{2}m\angle GHE$
- f) $m\angle x = m\angle y$
- g) $\angle x \cong \angle y$

3. Given: $\overleftrightarrow{ECAD}$
 $\angle x \cong \angle y$

Prove: $m\angle w + m\angle z = 180$

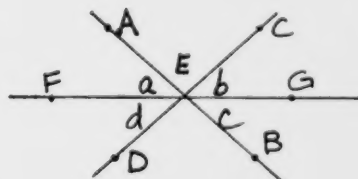
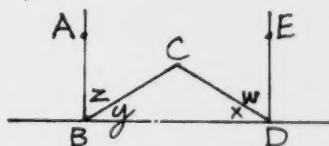


4. Given: $\overline{AB} \perp \overline{BD}$, $\overline{ED} \perp \overline{BD}$
 $\angle x \cong \angle y$

Prove: $\angle w \cong \angle z$

5. Given: $\overleftrightarrow{AEB}$, $\overleftrightarrow{DEC}$, $\overleftrightarrow{FEG}$
 $\angle a \cong \angle b$

Prove: $\angle c \cong \angle d$



HOMEWORK SOLUTIONS:

1. When $n = 41$, $P = 41^2 - 41 + 41 = 41^2$, which is divisible by 41.

2. a) Given
 b) Perpendicular lines form right angles.
 c) All right angles are congruent.
 d) Given
 e) A bisector divides an angle in half
 f) Halves of equal quantities are equal.
 g) If the measures of two angles are equal, then the angles are congruent.
3. a) $\overleftrightarrow{ECAD}$ a) Given
 b) $\angle x \cong \angle y$ b) Given
 c) $\angle y \cong \angle z$ c) Vertical $\angle$'s are congruent
 d) $\angle x \cong \angle z$ d) Transitivity
 e) $\angle x$ and $\angle w$ are supplementary e) If 2 $\angle$'s form a linear pair, then they are suppl.
 f) $\angle z$ and $\angle w$ are suppl. f) Substitution
 g) $m\angle z + m\angle w = 180$ f) If 2 $\angle$'s are supplementary the sum of their measures is 180
4. a) $\overline{AB} \perp \overline{BD}, \overline{ED} \perp \overline{BD}$ a) Given
 b) $\angle ABD$ and $\angle EDB$ are right angles b) Perpendicular lines form right angles
 c) $\angle ABD \cong \angle EDB$ c) All right $\angle$'s are $\cong$
 d) $\angle x \cong \angle y$ d) Given
 e) $\angle w \cong \angle z$ e) Subtraction Axiom
5. a) $\overleftrightarrow{AEB}, \overleftrightarrow{DEC}, \overleftrightarrow{FEG}$ a) Given
 b) $\angle AED \cong \angle CEB$ b) Vertical $\angle$'s are $\cong$
 c) $\angle a \cong \angle b$ c) Given
 d) $\angle d \cong \angle c$ d) Subtraction Axiom

Alternate proof for 5:

- a) $\overleftrightarrow{AEB}, \overleftrightarrow{DEC}, \overleftrightarrow{FEG}$ a) Given
 b) $\angle a \cong \angle c,$ b) Vertical angles are $\cong$
 $\angle b \cong \angle d$
 c) $\angle a \cong \angle b$ c) Given
 d) $\angle c \cong \angle d$ d) Transitive property

Note: In problems 2, 3, 4, 5 you may want to add steps involving the Partition Postulate.

Lesson 23 - Congruent Polygons

AIM: What are congruent polygons?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define congruent polygons.
2. Show that congruence of polygons is an equivalence relation.
3. Demonstrate that two given triangles are congruent using the "SAS $\cong$ SAS" postulate.

VOCABULARY: correspondence, included angle

MOTIVATION:

Johnny decided to construct an open book case for his room.
Is this a "strong" shape?



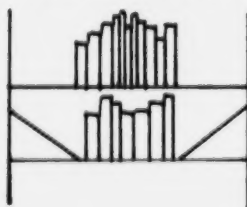
DEVELOPMENT:

1. Discuss the motivation problem. Demonstrate that a quadrilateral is not a rigid figure by showing a model of a quadrilateral. There are many models available in which the sides are metal bars which are fastened at the vertices. If a model is not available, it may be easily constructed out of cardboard and fasteners. Change the shape of the quadrilateral from a rectangle to an oblique parallelogram. Have students observe that even though the corresponding sides are congruent, the quadrilaterals have different shapes.

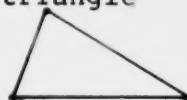
Inquire, "How could the bookcase be made rigid?"

Students should be able to suggest making a cross-bar.

Discuss various possibilities:



Ask, "What is it about the triangle that makes it useful in this construction?" Elicit that the triangle is a rigid figure. Demonstrate the rigidity of the triangle with a model. If the vertices are fastened, the triangle will not change its shape. It is this aspect of triangles that we will be studying in this unit.



Refer to applications of the rigidity of triangles in architecture. The most spectacular examples are in the Brooklyn Bridge, the Statue of Liberty and the geodesic dome. In his book, "The Great Bridge", John McCullough writes that the engineer Roebling used the triangle in supports for the Brooklyn Bridge.

2. Present a problem in manufacturing: A manufacturer wishes to mass produce a diamond-shaped emblem that will be sewn on shirts.



If two emblems were exactly the same size and shape, we would call them congruent. How could you tell if two diamonds are congruent?



(Place one diamond on top of the other.) How would you construct congruent emblems? Distribute a rexo containing a drawing of the emblem. Ask each student to use a ruler and protractor to duplicate the emblem. After some experimentation students will begin to copy sides and angles. From this experience have the class define congruent polygons in terms equivalent to the definition: Congruent polygons are polygons whose vertices can be paired so that corresponding sides and angles are congruent.

3. Do application #1.
4. Investigate correspondence more fully. Adopt the notation that if quadrilateral ABCD is congruent to quadrilateral EFGH, then $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FG}$, $\overline{CD} \cong \overline{GH}$, $\overline{DA} \cong \overline{HE}$, and $\angle A \cong \angle E$, $\angle B \cong \angle F$, $\angle C \cong \angle G$, $\angle D \cong \angle H$.
5. Do application #2.
6. Let us examine what type of relation congruence of polygons is.

Is it reflexive? Have students explain that a polygon is congruent to itself.

Is it symmetric? Have a student explain that if $ABCD = EFGH$, then $EFGH = ABCD$.

Is it transitive? Illustrate.

What do we call a relation that has these three properties? Elicit "equivalence relation" from the class.

7. Do application #3.

8. Say, "According to the definition, in order to prove that two triangles are congruent, how many parts do we have to prove congruent? (6) Yet if we are dealing with triangles it may not be necessary to know about all six parts. If I told you that I am thinking of a triangle one of whose sides is 6", could you draw the triangle?" Have students show different triangles, each with a 6" side.

Say, "If I told you that two sides were 6" and 8", and an angle's measure is 50° , could you draw the triangle using a ruler and protractor?" Have students show some different triangles and some congruent triangles. An overhead projector would be very useful in demonstrating congruent triangles.

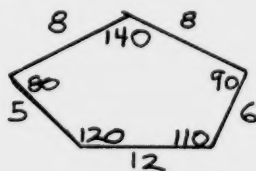
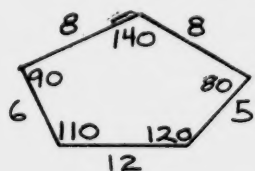
Elicit from the class that if the triangles agree in SAS, then they are congruent. State the postulate: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

Agree on the SAS notation, but insist on stating the postulate orally when the notation is used.

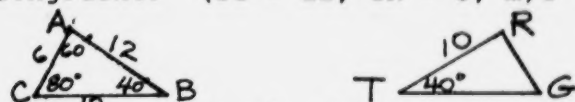
9. Do application #4.

APPLICATIONS:

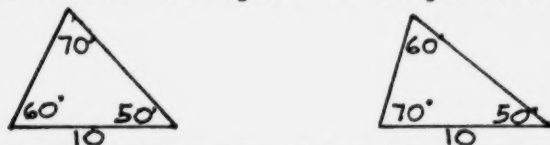
1. a) Are the polygons congruent? (yes)



- b) What would have to be true to make the triangles congruent? ($TG = 12$, $GR = 6$, $m\angle G = 60$, $m\angle R = 80$)



- c) Could the triangles be congruent? (no)

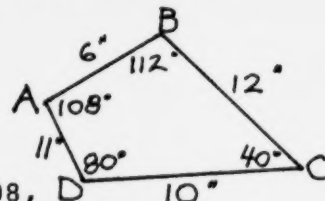


- d) How many ways can there be a correspondence between $\triangle ABC$ and $\triangle DEF$? (6 possibilities, depending on labeling)

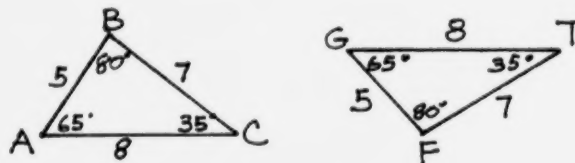
2. a) $ABCD \cong EFGH$

Find: FG , HE
 $m\angle E$
 $m\angle G$

(Answers:
 $FG = BC = 12$,
 $HE = DA = 11$,
 $m\angle E = m\angle A = 108$,
 $m\angle G = m\angle C = 40$)



- b) How would you state that the triangles are congruent using the $\cong$ notation? ($\triangle ABC \cong \triangle GFT$)



3. Is the relation "is the complement" an equivalence relation? (no, not reflexive or transitive)
 Is congruence of angles an equivalence relation? (yes)
 4. Are the triangles congruent by SAS?

- a)



b)



- c)



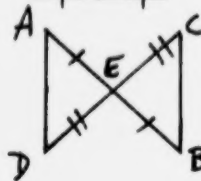
d)



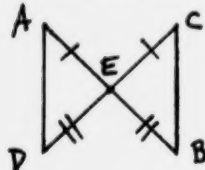
- e)



f)



- g)

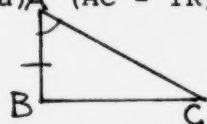


(Answers: a) yes, b) yes, c) no, not SAS, d) yes, e) no, not SAS, f) yes, g) yes)

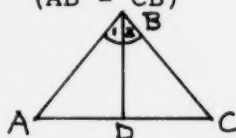
5. a) In the last two examples, 4f and 4g, both sets of triangles are congruent by SAS, but there is a difference in the congruence. What is it? (different parts are congruent)
- b) In 4f: Which side is $\cong \overline{AD}$? Which angle is $\cong \angle A$? ($\overline{BC}$, $\angle B$)
- In 4g: Which side is $\cong \overline{AD}$? Which angle is $\cong \angle A$? ($\overline{CB}$, $\angle C$)

6. In each diagram, what additional information would you need to use SAS to prove the triangles congruent?

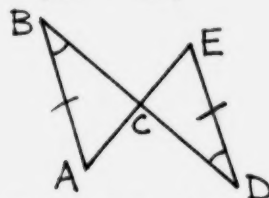
a) $(\overline{AC} \cong \overline{TR})$



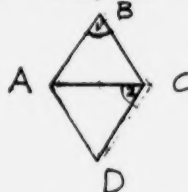
b) $(\overline{AB} \cong \overline{CB})$



c) $(\overline{BC} \cong \overline{DC})$

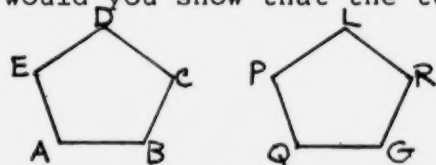


d) $(\overline{AB} \cong \overline{AC}, \overline{BC} \cong \overline{CD})$

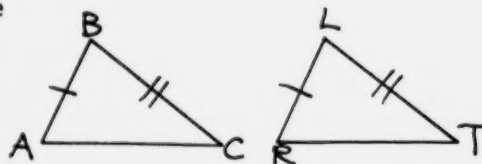


SUMMARY:

1. How would you show that the two pentagons are congruent?

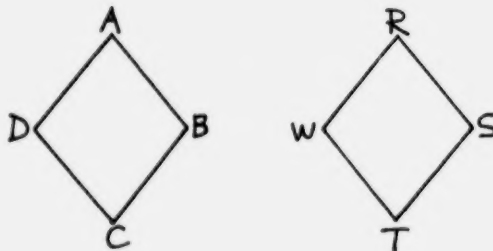


2. Is the information sufficient to say that the triangles are congruent? What else would you need? ($\angle B \cong \angle L$)



HOMEWORK:

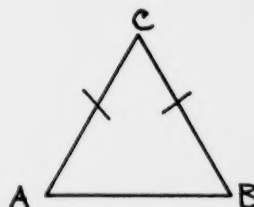
1. If $ABCD \cong RWTS$
Which angle is $\cong \angle B$?
Which side is $\cong \overline{DC}$?



2. If $\triangle ACB \cong \triangle BCA$

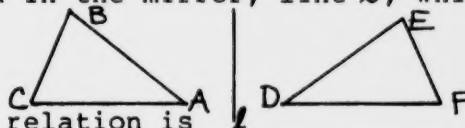
Which angle is $\cong \angle A$?

Which angle is $\cong \angle B$?



3. Write all the congruences of equilateral $\triangle ABC$ and itself.

4. If $\triangle ABC$ is reflected in the mirror, line ℓ , which parts would be congruent?

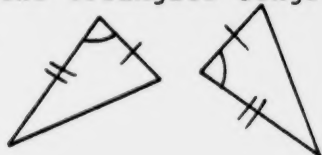


5. Indicate whether the relation is

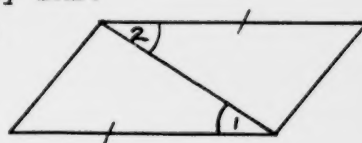
- i) reflexive ii) symmetric iii) transitive
a) "is congruent to" (for line segments)
b) "is the brother of"
c) "is the supplement of"

6. Are the triangles congruent by SAS?

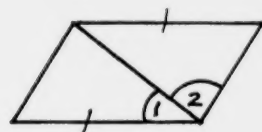
a)



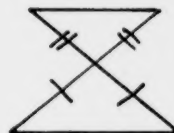
b)



c)



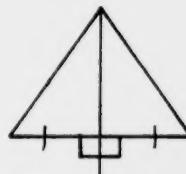
d)



e)

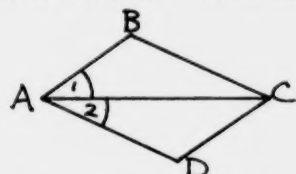


f)

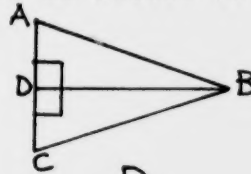


7. What additional information would be needed to use SAS?

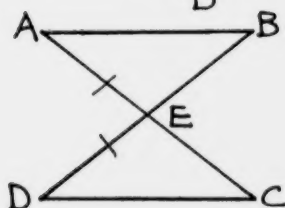
a)



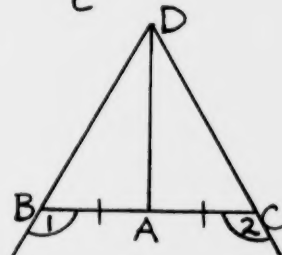
b)



c)



d)



HOMEWORK SOLUTIONS:

1. $\angle W \cong \angle B$

$\overline{DC} \cong \overline{ST}$

2. $\angle B \cong \angle A$

$\angle A \cong \angle B$

3. $\triangle ABC \cong \triangle ABC$

$\triangle ABC \cong \triangle ACB \cong \triangle BAC \cong \triangle BCA \cong \triangle CAB \cong \triangle CBA$

4. $\angle C \cong \angle F, \angle B \cong \angle E, \angle A \cong \angle D$

$\overline{BC} \cong \overline{EF}, \overline{BA} \cong \overline{ED}, \overline{CA} \cong \overline{FD}$

5. a) i) yes ii) yes iii) yes (Note: If A
 b) i) no ii) no iii) yes is the brother
 c) i) no ii) yes iii) no of B, B may be
 the sister of
 A).

6. a) yes b) yes c) no d) no e) yes f) yes

7. a) $\overline{AB} \cong \overline{AD}$

b) $\overline{AD} \cong \overline{CD}$

c) $\overline{BE} \cong \overline{CE}$

d) $\overline{BD} \cong \overline{CD}$

Lesson 24 - SAS Proofs

AIM: How do we use the "S.A.S. $\cong$ S.A.S." postulate to prove that two triangles are congruent?

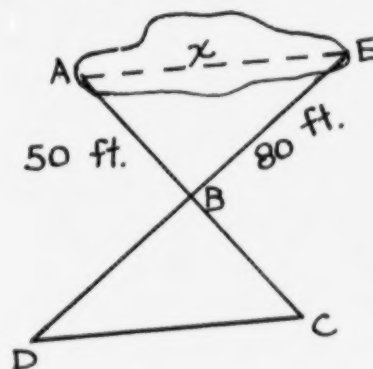
PERFORMANCE OBJECTIVES: The student will be able to...

1. Plan and write a formal proof that two triangles are congruent using the "S.A.S. $\cong$ S.A.S." postulate.

VOCABULARY: corollary

MOTIVATION:

How would you find the distance across a lake by staying on land and using this hint?



DEVELOPMENT:

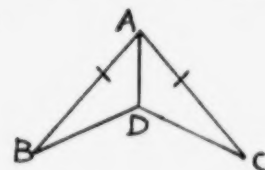
1. Discuss the motivation problem. Have students explain how they would make the triangles congruent by making $\overline{AB} \cong \overline{CB}$ and $\overline{EB} \cong \overline{DB}$. Have them justify the congruence by the SAS postulate identifying congruent parts. They should finally indicate that once the triangles are congruent, then the corresponding sides, $\overline{AE}$ and $\overline{CD}$ must be congruent.

Ask, "Could you have made $\overline{CB} = \overline{EB} = 80$ and $\overline{DB} = \overline{AB} = 50$? Would that have made the triangles congruent? Which sides would correspond? Which angles would correspond?" (yes; $\overline{CB} \cong \overline{EB}$, $\overline{DB} \cong \overline{AB}$, $\overline{DC} \cong \overline{AE}$; $\angle C \cong \angle E$, $\angle D \cong \angle A$, $\angle DBC \cong \angle ABE$)

2. In order to see the corresponding parts we must lay out the steps systematically in a proof.

Let's consider this diagram.

If $\overline{AB} = \overline{AC}$, would the triangles be congruent? (no) How do you know that they need not be congruent?



(not $\cong$ unless $\angle BAD \cong \angle CAD$)

Suppose $\overline{AB} \cong \overline{AC}$ and $\angle 1 \cong \angle 2$,

would the triangles be

congruent? (yes) How do we know? (SAS)

Let us write a proof.

Given: $\overline{AB} \cong \overline{AC}$, $\angle 1 \cong \angle 2$

Prove: $\triangle ABD \cong \triangle ACD$

How would you state the congruence of triangles?

Elicit: $\triangle ABD \cong \triangle ACD$ making $\overline{AB} \cong \overline{AC}$, etc.

Label the congruent given parts in the diagram. In the proof state the parts in the order S, A, S and indicate each side with an (S) in the proof and the angle with an (A):

- | | |
|--|----------------------------------|
| 1) $\overline{AB} \cong \overline{AC}$ (S) | 1) Given |
| 2) $\angle 1 \cong \angle 2$ (A) | 2) Given |
| 3) $\overline{AD} \cong \overline{AD}$ (S) | 3) Reflexive property of $\cong$ |
| 4) $\triangle ABD \cong \triangle ACD$ | 4) SAS = SAS |

Once the triangles are proven congruent, ask which other angles must be congruent. ($\angle B \cong \angle C$, $\angle ADB \cong \angle ADC$)

3. Do application #1.

4. Is this statement true? Two right triangles are congruent if the two legs of one are congruent to the two legs of the other. (yes) How do you know? (SAS, all right $\angle$ s are $\cong$)

Write out the proof.

A theorem that follows immediately from another statement is called a corollary.

5. Do application #2.

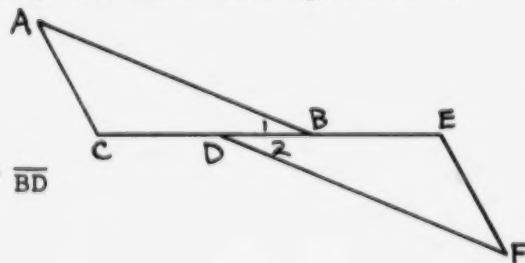
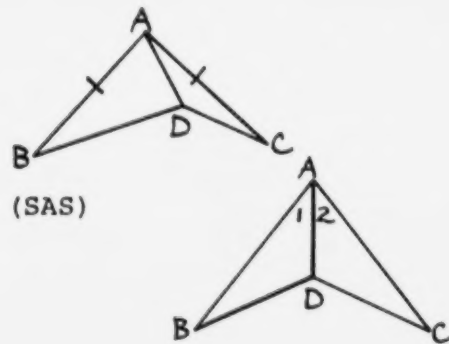
6. In some problems we may not be given an entire part of a triangle.

Given: $\overline{AB} \cong \overline{FD}$, $\overline{BE} \cong \overline{DC}$

Prove: $\angle 1 \cong \angle 2$
 $\triangle ABC \cong \triangle FDE$

(use addition axiom with $\overline{DB} \cong \overline{BD}$
 by reflexive property)

7. Do application #3.



8. If the class appears to have mastered the above material introduce problem (a). If not, reinforce with problem (b).

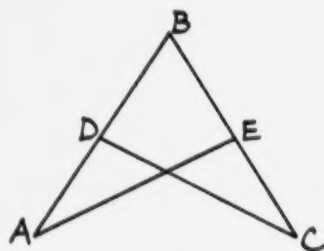
a) Given: $\overline{AB} \cong \overline{CB}$

$\overline{DB} \cong \overline{EB}$

Prove: $\triangle ABE \cong \triangle CBD$

(use $\angle B \cong \angle B$ by reflexive property of $\cong$)

Do application #4a.



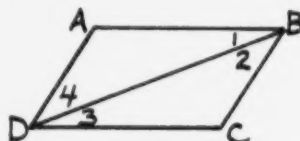
b) Given: $\overline{AB} \cong \overline{CD}$

$\angle 1 \cong \angle 3$

Prove: $\triangle ABD \cong \triangle CDB$

(use $\overline{BD} \cong \overline{DB}$ by reflexive property of $\cong$)

Do application #4b.

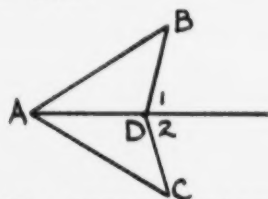


APPLICATIONS:

1. Given: $\overline{DB} \cong \overline{DC}$

$\angle 1 \cong \angle 2$

Prove: $\triangle ABD \cong \triangle ACD$ (use supplements of $\cong$ angles are $\cong$ and $\overline{AD} \cong \overline{AD}$ by reflexive property of $\cong$)



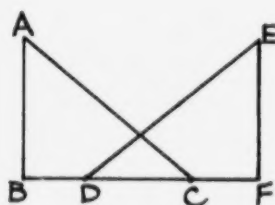
2. Prove: Two isosceles triangles are congruent if one leg and the vertex angle of one triangle are congruent to the corresponding parts of the other. (the other legs are also $\cong$, SAS)

3. Given: $\overline{AB} \cong \overline{EF}$, $\overline{AB} \perp \overline{BF}$

$\overline{EF} \perp \overline{FB}$, $\overline{BD} \cong \overline{FC}$

Prove: $\triangle ABC \cong \triangle EFD$

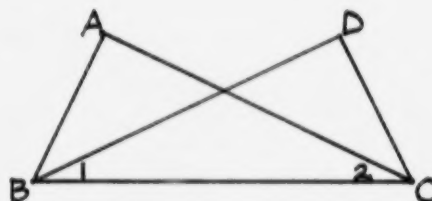
(use $\perp$ s form right $\angle$ s and all right $\angle$ s are $\cong$ reflexive property of $\cong$ and addition axiom)



4. a) Given: $\overline{AC} \cong \overline{DB}$

$\angle 1 \cong \angle 2$

Prove: $\triangle ABC \cong \triangle DCB$



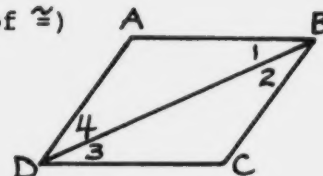
(use $\overline{BC} \cong \overline{CB}$ by reflexive property of $\cong$)

b) Given: $\overline{AB} \cong \overline{CB}$

$\angle 1 \cong \angle 2$

Prove: $\triangle ABD \cong \triangle CBD$

(use $\overline{BD} \cong \overline{DB}$ by reflexive property of $\cong$)



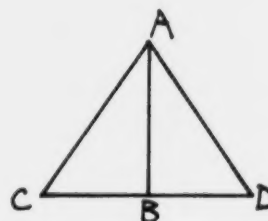
SUMMARY:

1. What is the meaning of $SAS \cong SAS$?
2. In order to use SAS in the figure below, what additional information do you need? ($\angle 2 \cong \angle 3$)

HOMEWORK:

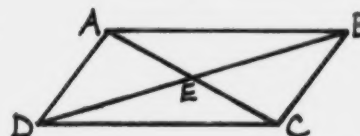
1. Given: $\overline{AB}$ is the $\perp$ bisector of $\overline{CD}$.

Prove: $\triangle CAB \cong \triangle DAB$



2. Given: $\overline{AE} \cong \overline{CE}$, $\overline{DE} \cong \overline{BE}$

Prove: $\triangle DAE \cong \triangle BCE$



3. Prove: Two right isosceles triangles are congruent if one leg of one triangle is congruent to a leg of the other.

4. Given: $\overline{AD} \cong \overline{CE}$, $\overline{DB} \cong \overline{EB}$

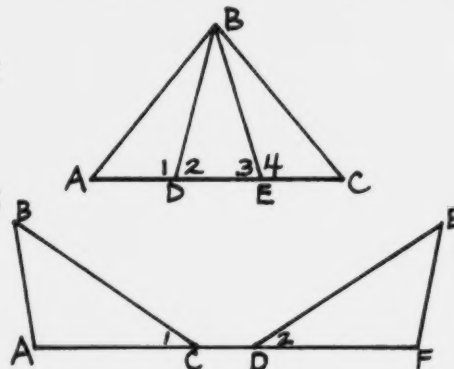
$\angle 2 \cong \angle 3$

Prove: $\triangle ABD \cong \triangle CBE$

5. Given: $\overline{AD} \cong \overline{FC}$, $\overline{BC} \cong \overline{ED}$

$\angle 1 \cong \angle 2$

Prove: $\triangle ABC \cong \triangle FED$



HOMEWORK SOLUTIONS:

1. 1) $\overline{AB} \cong \overline{AB}$ (S)
2) $\overline{AB}$ is the $\perp$ bisector of $\overline{CD}$

- 1) Reflexive Property of $\cong$
2) Given

3) $\angle ABC$ and $\angle ABD$ are right angles

4) $\angle ABC \cong \angle ABD$ (A)

5) $\overline{BC} \cong \overline{BD}$

6) $\triangle CAB \cong \triangle DAB$

3) Perpendicular lines form right angles

4) All right angles are congruent

5) A bisector divides a line segment into 2 $\cong$ segments

6) SAS $\cong$ SAS

2. 1) $\overline{AE} \cong \overline{CE}$ (S)

2) $\angle AED \cong \angle CEB$ (A)

3) $\overline{DE} \cong \overline{BE}$ (S)

4) $\triangle DAE \cong \triangle BCE$

1) Given

2) Vertical angles are $\cong$

3) Given

4) SAS $\cong$ SAS

3. Given: $\triangle ABC$ and $\triangle DEF$ are right isosceles triangles with right angles at B and E, $\overline{AB} \cong \overline{DE}$

Prove: $\triangle ABC \cong \triangle DEF$

1) $\triangle ABC$ and $\triangle DEF$ are right isosceles triangles

2) $\angle B$ and $\angle E$ are right angles

3) $\overline{AB} \cong \overline{DE}$ (S)

4) $\angle B \cong \angle E$ (A)

5) $\overline{AB} \cong \overline{BC}$

6) $\overline{DE} \cong \overline{BC}$

7) $\overline{DE} \cong \overline{EF}$

8) $\overline{BC} \cong \overline{EF}$ (S)

9) $\triangle ABC \cong \triangle DEF$

1) Given

2) Given

3) Given

4) All right $\angle$'s are $\cong$

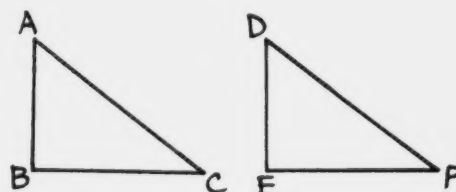
5) An isos. $\triangle$ has 2 $\cong$ legs

6) Transitive property of $\cong$

7) Same as 5)

8) Same as 6)

9) SAS $\cong$ SAS



4. 1) $\overline{AD} \cong \overline{CE}$ (S)

2) $\angle 2 = \angle 3$

3) $\angle 1$ suppl. $\angle 2$
 $\angle 4$ suppl. $\angle 3$

4) $\angle 1 \cong \angle 4$ (A)

5) $\overline{DB} \cong \overline{EB}$ (S)

6) $\triangle ABD \cong \triangle CBE$

1) Given

2) Given

3) If 2 adj. $\angle$'s have their non-adj. sides on the same straight line, then they are suppl.

4) Supplements of $\cong \angle$'s are $\cong$

5) Given

6) SAS $\cong$ SAS

5. 1) $\overline{AD} \cong \overline{FC}$
 so $\overline{AD} = \overline{FC}$
 2) $CD = DC$
 3) $AD - CD = FC - DC$
 or $AC = FD$
 4) $\overline{AC} \cong \overline{FD}$ (S)
- 5) $\angle 1 \cong \angle 2$ (A)
 6) $\overline{BC} = \overline{ED}$ (S)
 7) $\triangle ABC \cong \triangle FED$

- 1) Given
 2) Equality is reflexive.
 3) Subtraction Axiom
 4) If 2 line segments have
 the same lengths
 they are $\cong$.
 5) Given
 6) Given
 7) SAS $\cong$ SAS

Lesson 25 - ASA and SSS

AIM: How do we use the " $ASA \cong ASA$ " and " $SSS \cong SSS$ " postulates to prove that two triangles are congruent?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Mark a diagram and decide which congruence method is appropriate.
2. Plan a congruence proof.
3. Write a formal proof using the correct method of congruence.

VOCABULARY: included side

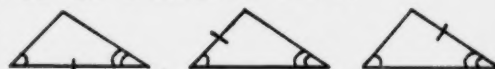
MOTIVATION:

Distribute a xeroxed sheet containing a printed triangle. Ask each student to copy the triangle using a straightedge and a compass. Allow them to use any method they can devise.

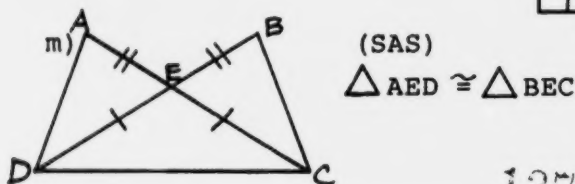
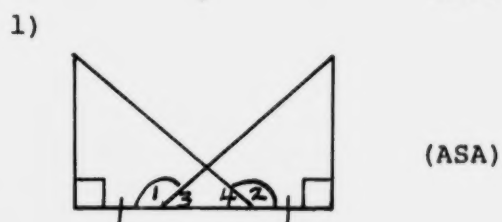
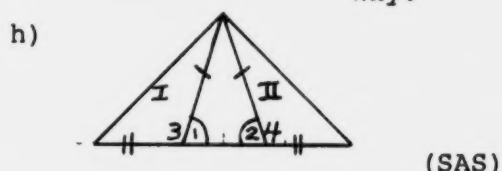
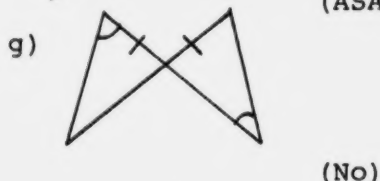
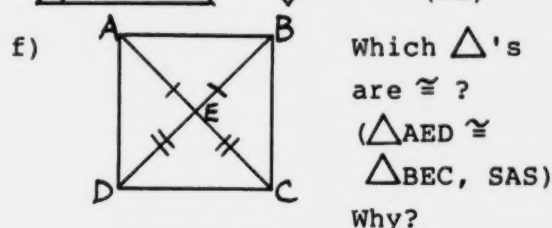
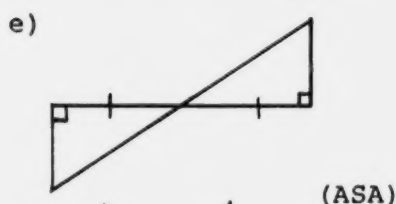
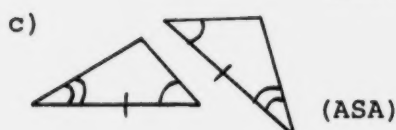
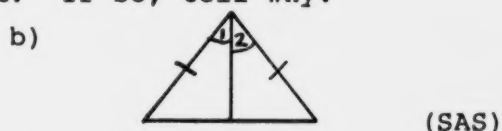
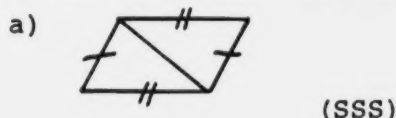
DEVELOPMENT:

1. Discuss the motivation problem. Ask the class, "Does anyone have a quick way to copy the triangle?" After a student explains his method, ask, "Does anyone have a quicker method?"
From the discussion it should become clear that copying three sides is the quickest method. Have a student state the SSS postulate. Indicate that when a triangle is called "rigid", it means that if the three sides are given, there is only one shape for the triangle. All triangles with three sides congruent to the sides of the given triangle must be congruent to it and have corresponding congruent angles.
2. If we were required to use a ruler and protractor instead of compasses, would you have used the same method to copy the triangle? (No, SSS is not a convenient method with

ruler and protractor.) What could we have done?
 Elicit the ASA postulate through their efforts at
 construction. Clarify the meaning of an included side.
 Distinguish among the three cases:



3. Now that we have three methods of establishing congruent triangles, we have to be clear in using the right method. Give the class a series of exercises in which the given parts are marked in the diagram and the students have to indicate which method of congruence can be used. (Grade the exercises according to the ability of the class.)
 Are the triangles congruent? If so, tell why:



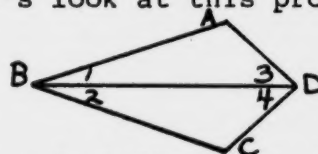
4. Have students make up their own diagrams and problems and present them to the class.

5. If we are asked to construct a formal proof of congruent triangles, the first step is to label the diagram and make a plan of attack. Let's look at this proof.

Given: $\angle 1 \cong \angle 2$

$\angle 3 \cong \angle 4$

Prove: $\triangle ABD \cong \triangle CBD$



Label the diagram using colored chalk. Emphasize the writing of a plan. Write: "Plan: Prove the triangles congruent by showing ASA $\cong$ ASA. The congruent sides are $\overline{BD} \cong \overline{BD}$."

Write all the steps of the proof. After the proof is written and discussed, ask, "What would happen in the problem if the given were changed to $\angle 1 \cong \angle 4$ and $\angle 2 \cong \angle 3$?" ($\triangle$'s still $\cong$ but with different corresponding parts)

6. Study the applications. In each example, after labeling the diagram, have the students write a plan and read it to the class.

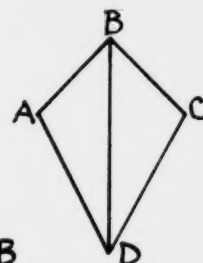
APPLICATIONS:

1. Given: $\overline{AB} \cong \overline{CB}$

$\overline{AD} \cong \overline{CD}$

Prove: $\triangle ABD \cong \triangle CBD$

(use $\overline{BD} \cong \overline{BD}$ by reflexive property, SSS)



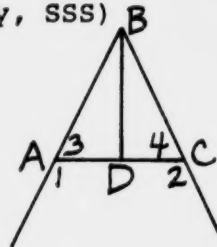
2. Given: $\angle 1 \cong \angle 2$

$\overline{DA} \cong \overline{DC}$, $\overline{AB} \cong \overline{CB}$

Prove: $\triangle ABD \cong \triangle CBD$

(use $\angle 3 \cong \angle 4$ by supp.

of $\cong$ $\angle$'s are $\cong$, SAS)

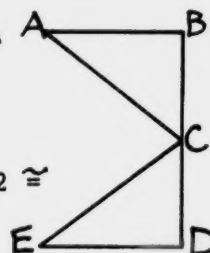


3. Given: C is the midpoint of $\overline{BD}$.

$\overline{AB} \cong \overline{ED}$, $\overline{AB} \perp \overline{BD}$, $\overline{ED} \perp \overline{DB}$

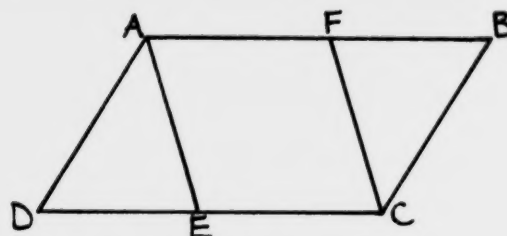
Prove: $\triangle ABC \cong \triangle EDC$

(use midpoint divides line segment into 2 $\cong$ parts, all right $\angle$'s are $\cong$, SAS)



4. Given: $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$, $\overline{AF} \cong \overline{CE}$

$\angle D \cong \angle B$
 Prove: $\triangle ADE \cong \triangle CBF$
 (use subtraction axiom, SAS)



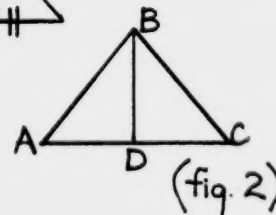
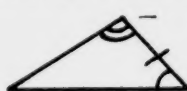
SUMMARY:

1. What congruence methods do we have for triangles?
 (SAS, ASA, SSS)
2. How would you prove
 two quadrilaterals
 congruent?
3. What are the steps in a formal proof of congruent
 triangles?



HOMEWORK:

1. Indicate whether or not the triangles are congruent. If
 so, write the method of congruence.

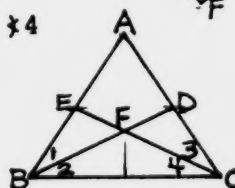
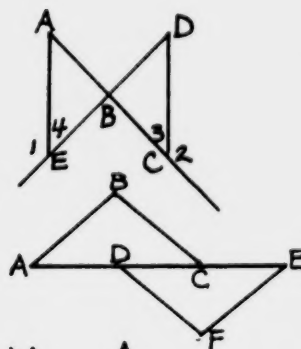


2. Given: $\overline{AB} = \overline{CB}$, $\overline{AD} = \overline{CD}$
 Prove: $\triangle ABD \cong \triangle CBD$

3. Given: $\angle 1 \cong \angle 2$
 $\overline{EB} \cong \overline{CB}$
 Prove: $\triangle AEB \cong \triangle DCB$

4. Given: $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FD}$
 $\overline{AD} \cong \overline{EC}$
 Prove: $\triangle ABC \cong \triangle EFD$

5. Given: $\angle ABC \cong \angle ACB$, $\angle 2 \cong \angle 4$
 $\overline{BF} \cong \overline{CF}$
 Prove: $\triangle BFE \cong \triangle CFD$



HOMEWORK SOLUTIONS:

- | | | |
|----|---|--|
| 1. | a) Yes (ASA $\cong$ ASA) | b) Yes (SAS $\cong$ SAS) |
| | c) No | d) Yes (SAS $\cong$ SAS) |
| 2. | 1) $\overline{AB} \cong \overline{CB}$ (S) | 1) Given |
| | 2) $\overline{AD} \cong \overline{CD}$ (S) | 2) Given |
| | 3) $\overline{BD} \cong \overline{BD}$ (S) | 3) Reflexive property of $\cong$ |
| | 4) $\triangle ABD \cong \triangle CBD$ | 4) SSS $\cong$ SSS |
| 3. | 1) $\angle ABE \cong \angle DBC$ (A) | 1) Vertical $\angle$'s are $\cong$ |
| | 2) $\overline{EB} \cong \overline{CB}$ (S) | 2) Given |
| | 3) $\angle 1 \cong \angle 2$ | 3) Given |
| | 4) $\angle 4$ suppl. $\angle 1$
$\angle 3$ suppl. $\angle 2$ | 4) If 2 adj. $\angle$'s have their non-adj. sides in a str. line, they are suppl. |
| | 5) $\angle 4 \cong \angle 3$ (A) | 5) Suppl. of $\cong \angle$'s are $\cong$ |
| | 6) $\triangle AEB \cong \triangle DCB$ | 6) ASA $\cong$ ASA |
| 4. | 1) $\overline{AB} \cong \overline{EF}$ (S) | 1) Given |
| | 2) $\overline{BC} \cong \overline{FD}$ (S) | 2) Given |
| | 3) $\overline{AD} \cong \overline{EC}$ | 3) Given |
| | 4) $\overline{DC} = \overline{CD}$ | 4) Reflexive property of = |
| | 5) $\overline{AD} + \overline{DC} =$
$\overline{EC} + \overline{CD}$ OR
$\overline{AC} = \overline{ED}$ | 5) Addition axiom |
| | 6) $\overline{AC} \cong \overline{ED}$ (S) | 6) If 2 line segments have the same length they are $\cong$. |
| | 7) $\triangle ABC \cong \triangle EFD$ | 7) SSS $\cong$ SSS |
| 5. | 1) $\angle ABC \cong \angle ACB$
so $m\angle ABC = m\angle ACB$ | 1) Given |
| | 2) $\angle 2 \cong \angle 4$
so $m\angle 2 = m\angle 4$ | 2) Given |
| | 3) $m\angle ABC - m\angle 2 =$
$m\angle ACB - m\angle 4$
or $m\angle 1 = m\angle 3$ | 3) Subtraction axiom |
| | 4) $\angle 1 \cong \angle 3$ (A) | 4) If two $\angle$ s have the same measure they are $\cong$. |
| | 5) $\overline{BF} \cong \overline{CF}$ (S) | 5) Given |
| | 6) $\angle BFE \cong \angle CFD$ (A) | 6) Vertical $\angle$ s are $\cong$ |
| | 7) $\triangle BFE = \triangle CFD$ | 7) ASA $\cong$ ASA |

Lesson 26 - More Difficult Congruent Triangles

AIM: How do we do a more difficult congruent triangles proof?

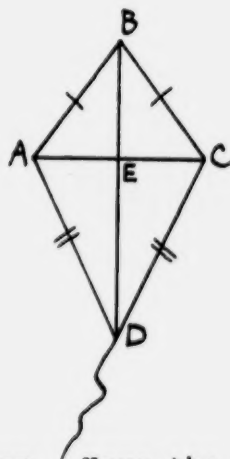
PERFORMANCE OBJECTIVES: The student will be able to...

1. Analyze a congruent triangles proof and write a plan.
2. Write a formal proof using the correct method of congruence.

VOCABULARY: conjecture

MOTIVATION:

Suzy noticed that her kite was made up of congruent triangles. How many sets can you name?



DEVELOPMENT:

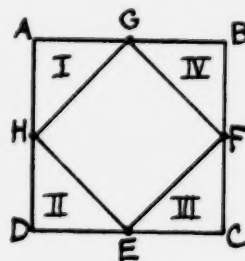
1. Discuss the motivation problem. Have the class simply list all the sets of congruent triangles. Then ask, "Can anyone prove that these two triangles are congruent?" Establish that you must first prove $\triangle ABD \cong \triangle CBD$ by showing $SSS \cong SSS$. Once those triangles are proved congruent, have students indicate the corresponding angles that are congruent. Using the corresponding congruent angles, have students show informally how $\triangle ABE \cong \triangle CBE$ and $\triangle ADE \cong \triangle CDE$ by $SAS \cong SAS$. You might also want to investigate informally why the diagonals are perpendicular in the kite.

2. Let's look at some other quadrilaterals.

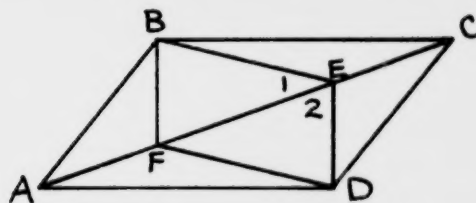
a) Given: ABCD is a square.

E, F, G, H are midpoints.

Prove: $\triangle I \cong \triangle II \cong \triangle III \cong \triangle IV$



- b) Given: $\overline{BE} \cong \overline{FD}$
 $\angle 1 \cong \angle 2$
 Prove: $\triangle BEF \cong \triangle DEF$
 $\triangle BCE \cong \triangle DCE$



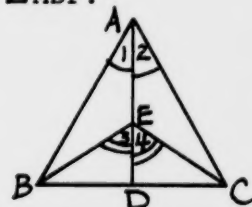
In each example, have the class label the diagram and write a plan. It is a good idea to extend a solved problem. Areas of investigation might be:

In problem a), what type of figure is EFGH?

In problem b), can you prove $\triangle ABF \cong \triangle ADF$?

3. "Prove as many things as you can about this diagram." Introduce the term "conjecture."

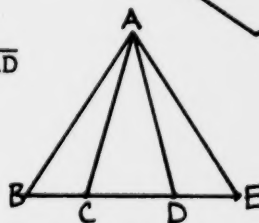
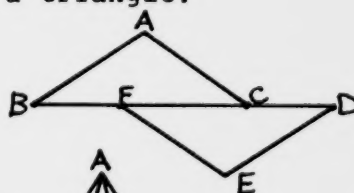
Given: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$



Ask, "Was anyone able to prove a set of triangles congruent? Which ones?" Write the set of triangles and the plan by which they could be proved congruent. After several attempts someone should suggest that $\triangle AEB$ and $\triangle AEC$ can be proved congruent by showing $ASA \cong ASA$. Write the steps of the proof. After the proof is explained, depending on the ability of the class, you may want to pursue other congruent triangles such as $\triangle BED \cong \triangle CED$ by SAS or $\triangle BAD \cong \triangle CAD$ by SAS or ASA.

4. Present problems in which segments have to be added or subtracted to obtain a side of a triangle.

- a) Given: $\angle B \cong \angle D$
 $\angle ACB \cong \angle EFD$
 $\overline{BF} \cong \overline{DC}$
 Prove: $\triangle ABC \cong \triangle EDF$
 b) Given: $\overline{BD} \cong \overline{EC}$, $\overline{AC} \cong \overline{AD}$
 $\overline{AB} \cong \overline{AE}$
 Prove: $\triangle BAC \cong \triangle EAD$



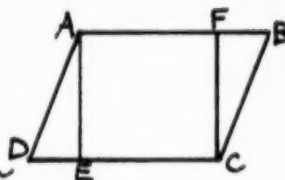
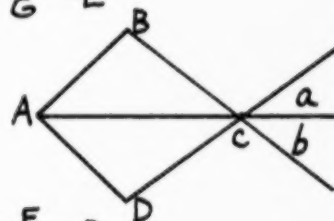
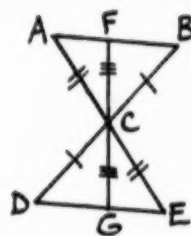
5. Study verbal problems:

If ABCD is a square and M is the midpoint of $\overline{AB}$, prove $\triangle MAD \cong \triangle MBC$.

Help students to draw an appropriate diagram, write a plan, and write the proof.

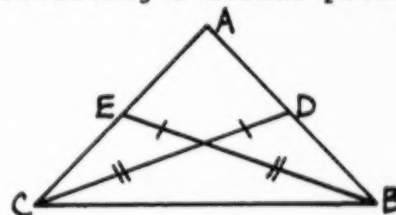
APPLICATIONS:

1. Prove as many pairs of triangles congruent as you can. (use vertical angles are congruent and prove triangles $\cong$ by SAS)
2. Given: $\overline{BC} \cong \overline{DC}$
 $\angle a \cong \angle b$
 Prove: $\triangle ABC \cong \triangle ADC$
 (use vertical angles are $\cong$, SAS)
3. Given: $\angle D \cong \angle B$
 $\overline{AE} \perp \overline{CD}$, $\overline{CF} \perp \overline{AB}$
 $\overline{AB} \cong \overline{CD}$, $\overline{AF} \cong \overline{CE}$
 Prove: $\triangle ADE \cong \triangle CBF$
 (use all right angles are $\cong$, subtraction axiom, ASA)
4. Prove: The median drawn from the vertex of an isosceles triangle divides the triangle into two congruent triangles. (Use definition of median, definition of isosceles $\triangle$, SSS)



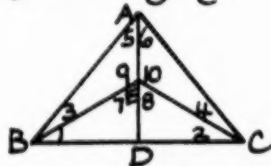
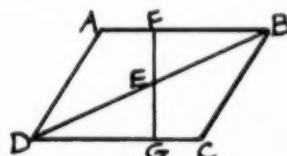
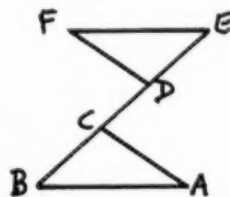
SUMMARY:

1. What is the procedure for constructing a formal proof of congruent triangles?
2. Determine a set of congruent triangles and plan the proof of congruence.



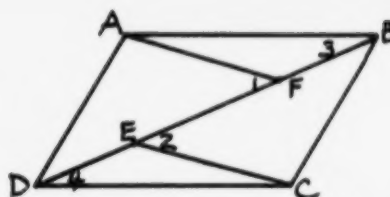
HOMEWORK:

1. Given: $\overline{EC} \cong \overline{BD}$, $\overline{AB} \cong \overline{FE}$
 $\angle E \cong \angle B$
 Prove: $\triangle ABC \cong \triangle FED$
2. Prove as many sets of triangles congruent as you can.
 - a) Given: $\overline{AB} \cong \overline{CD}$, $\overline{CB} \cong \overline{AD}$
 $\overline{FG}$ bisects $\overline{BD}$.
 - b) Given: $\angle 5 \cong \angle 6$
 $\angle 9 \cong \angle 10$



3. Given: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$
 $\overline{DF} \cong \overline{BE}$

Prove: $\triangle CED \cong \triangle AFB$



4. Prove: The bisector of the vertex angle of an isosceles triangle forms two congruent triangles.

HOMEWORK SOLUTIONS:

1. 1) $\overline{EC} \cong \overline{BD}$ so
 $EC = BD$
 2) $DC = DC$
 3) $EC - DC = BD - DC$
 or $ED = BC$

4) $\overline{ED} \cong \overline{BC}$ (S)

5) $\angle E \cong \angle B$ (A)

6) $\overline{AB} \cong \overline{FE}$ (S)

7) $\triangle ABC \cong \triangle FED$

- 2a. 1) $\overline{AB} \cong \overline{CD}$ (S)
 2) $\overline{CB} \cong \overline{AD}$ (S)
 3) $\overline{BD} \cong \overline{BD}$ (S)
 4) $\triangle ABD \cong \triangle CDB$
 5) $\angle FBE \cong \angle GDE$ (A)
 6) $\overline{FG}$ bisects $\overline{BD}$
 7) $\overline{BE} \cong \overline{DE}$ (S)

8) $\angle FEB \cong \angle GED$ (A)

9) $\triangle FEB \cong \triangle GED$

- 2b. 1) $\angle 5 \cong \angle 6$ (A)
 2) $\overline{AE} \cong \overline{AE}$ (S)
 3) $\angle 9 \cong \angle 10$ (A)
 4) $\triangle ABE \cong \triangle ACE$
 5) $\overline{EB} \cong \overline{EC}$ (S)
 6) $\angle 7$ suppl. $\angle 9$
 $\angle 8$ suppl. $\angle 10$

7) $\angle 7 \cong \angle 8$ (A)

1) Given

2) Reflexive prop.

3) Subtraction princ.

4) If 2 line segments have the same measurer they are $\cong$.

5) Given

6) Given

7) SAS $\cong$ SAS

1) Given

2) Given

3) Reflexive prop.

4) SSS $\cong$ SSS

5) Corr. $\angle$'s of $\cong \triangle$ s are $\cong$

6) Given

7) To bisect is to divide into 2 $\cong$ parts.

8) Vertical $\angle$'s are $\cong$

9) ASA $\cong$ ASA

1) Given

2) Given

3) Given

4) ASA $\cong$ ASA

5) Corr. sides of $\cong \triangle$ s are $\cong$

6) If 2 adj. $\angle$'s have their non-adj. sides in a str. line, they are suppl.

7) Suppl. of $\cong \angle$'s are $\cong$

$$8) \overline{ED} \cong \overline{ED} \text{ (S)}$$

$$9) \triangle BED \cong \triangle CED$$

$$10) \overline{BA} \cong \overline{CA} \text{ (S)}$$

$$11) \overline{AD} \cong \overline{AD} \text{ (S)}$$

$$12) \triangle BAD \cong \triangle CAD$$

3. 1) $\angle 1 \cong \angle 2$
 2) $\angle AFB$ suppl. $\angle 1$
 $\angle CED$ suppl. $\angle 2$

$$3) \angle AFB \cong \angle CED \text{ (A)}$$

$$4) \overline{DF} \cong \overline{BE}$$

$$5) EF = EF$$

$$6) DF - EF = BE - EF$$

or $DE = BF$

$$7) \overline{DE} \cong \overline{BF} \text{ (S)}$$

$$8) \angle 3 \cong \angle 4 \text{ (A)}$$

$$9) \triangle CED \cong \triangle AFB$$

4. Given: $\overline{AB} \cong \overline{CB}$

$\overline{BD}$ is an $\angle$ bisector

Prove: $\triangle ABD \cong \triangle CBD$

$$1) \overline{AB} \cong \overline{CB} \text{ (S)}$$

$$2) \overline{BD} \text{ is an } \angle \text{ bisector}$$

$$3) \angle 1 \cong \angle 2 \text{ (A)}$$

$$4) \overline{BD} \cong \overline{BD} \text{ (S)}$$

$$5) \triangle ABD \cong \triangle CBD$$

8) Reflexive prop. of $\cong$

$$9) SAS \cong SAS$$

10) Corr. sides of $\cong \Delta$'s are $\cong$
 From step 4.

11) Reflexive prop. of $\cong$

$$12) SAS \cong SAS$$

1) Given

2) If 2 adj. $\angle$'s have their non-adj. sides in a str. line, they are suppl.

3) Suppl. of $\cong \angle$'s are $\cong$

4) Given

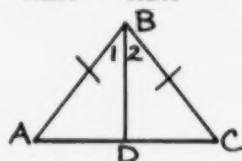
5) Reflexive prop. of equality

6) Subtraction princ.

7) If 2 line segments have the same length, they are $\cong$

8) Given

$$9) ASA \cong ASA$$



1) Given

2) Given

3) To bisect is to divide into 2 $\cong$ parts

4) Reflexive prop.

$$5) SAS \cong SAS$$

Lesson 27 - Corresponding Parts of Congruent Triangles

AIM: How do we use congruent triangles to prove line segments or angles congruent?

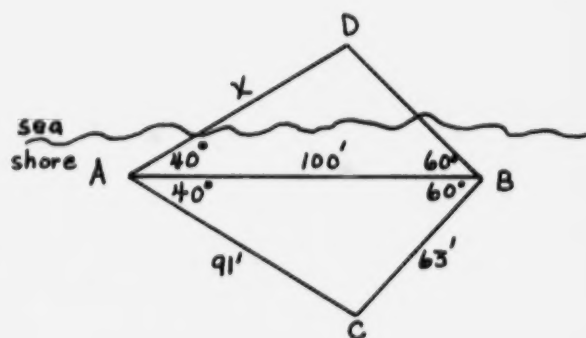
PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove line segments or angles are congruent by
 - a) correctly choosing a pair of triangles containing the angles or line segments.
 - b) choosing the correct congruence method.
 - c) using "corresponding parts of congruent triangles are congruent".
 - d) writing out the proof after performing the above analysis.

VOCABULARY: corresponding parts

MOTIVATION:

A boy on the beach wanted to find out how far he was from a boat at sea. He measured the angle at A and found it to be 40° while his friend at B, 100 feet away, measured $\angle B$ and found it to be 60° . Next, they walked away from the shore at angles equal to the ones they had just measured. They measured the distances to the point where they met, C. AC was 91 feet and BC was 63 feet. Find AD.



DEVELOPMENT:

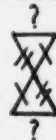
1. Discuss the motivation problem. Ask the students how they found the answers. Elicit that the triangles are congruent by ASA and that corresponding parts are congruent. Make it clear why $AD = 91$ and not 63 by identifying all of the corresponding parts. Have the

class state and write: Corresponding parts of congruent triangles are congruent.

2. A student noticed that if two segments bisect each other, the segments joining the ends of the given segments are congruent. How would you prove this conjecture?

Label the diagram accordingly.

Use a question mark to label the parts to be proven congruent.



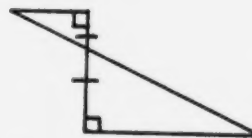
Have the students write a plan and then construct the proof. (vertical angles, SAS) Ask, "What other parts are congruent?" (Corresponding sides and angles - have students name them) After the problem is solved, change the information.

Suppose we had drawn the diagram this way:

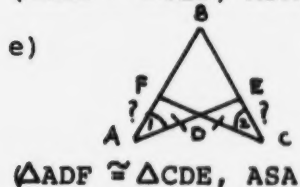
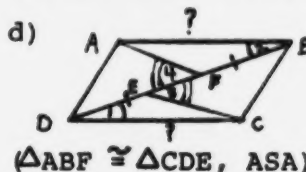
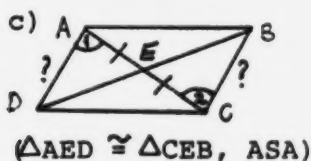
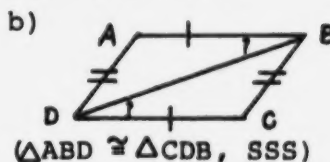
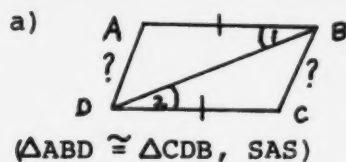


Triangles are not congruent. Why not?

Which angles are congruent?



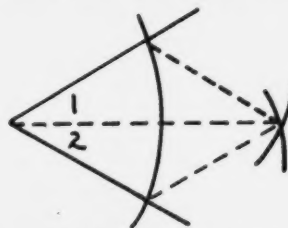
3. Present a series of examples in which the class has to write only the plan. "In order to prove the parts labeled are congruent, which triangles would you prove congruent and what method would you use?"



4. Choose one of the above in (3) and write a complete proof.

5. Show how congruent triangles can be used to justify the constructions. Have the class bisect an angle using a straightedge and compasses. Ask, "How do we know the construction is correct?"

Have the class analyze each step of the construction and determine the method of congruence. (SSS)



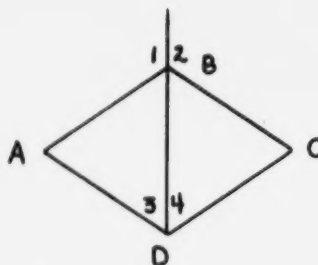
6. Give the class practice with verbal problems. Prove: If a point is on the perpendicular bisector of a line segment, it is equidistant from the ends of the line segment. ($\cong \Delta$ s by SAS)

APPLICATIONS:

1. Given: $\overline{AB} \cong \overline{CB}$

$$\angle 1 \cong \angle 2$$

Prove: $\angle 3 \cong \angle 4$
(supp. of $\cong$ $\angle$'s
 $\cong$, SAS)



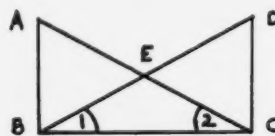
2. Given: $\angle 1 \cong \angle 2$

$$\overline{AB} \perp \overline{BC}, \overline{DC} \perp \overline{CB}$$

$$\overline{BE} \cong \overline{CE}$$

Prove: $\overline{AB} \cong \overline{DC}$

(overlapping Δ 's ABC and DCB
in better classes; otherwise,
comp. of $\cong$ $\angle$'s are $\cong$ and vertical
 $\angle$'s are $\cong$)



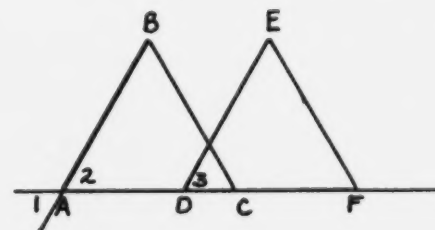
3. Show that the method of erecting a perpendicular from a point on a line is valid.
4. Prove: The bisector at the vertex angle of an isosceles triangle bisects the base. (SAS)

5. Given: $\angle 1 \cong \angle 3$

$$\overline{AB} \cong \overline{DE}, \overline{AD} \cong \overline{CF}$$

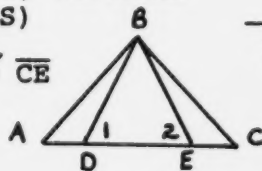
Prove: $\overline{BC} \cong \overline{EF}$

(addition post, vertical
 $\angle$'s are $\cong$, SAS)



6. Given: $\overline{BD} \cong \overline{BE}, \overline{AD} \cong \overline{CE}$

$$\angle A \cong \angle C$$



Prove: $\angle 1 = \angle 2$
(SAS, then supp. of $\angle$'s are $=$)

SUMMARY:

1. What are the methods we have of proving line segments congruent?
2. What are the methods we have of proving angles congruent?

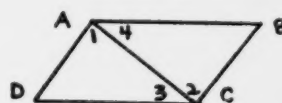
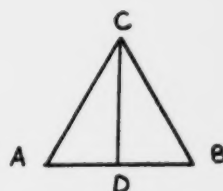
HOMEWORK:

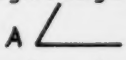
1. Given: $\overline{CD}$ bisects $\angle ACB$
 $\overline{CD} \perp \overline{AB}$

Prove: $\overline{AC} \cong \overline{BC}$

2. Given: $\angle 1 \cong \angle 2$
 $\overline{AD} \cong \overline{CB}$

Prove: $\angle 3 \cong \angle 4$

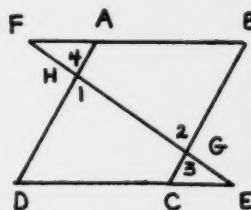


3. a) Copy $\angle A$ using straightedge and compasses. 
b) Prove your construction is correct by using congruent triangles.

4. Prove the median to the base of an isosceles triangle bisects the vertex angle.

5. Given: $\angle 3 \cong \angle 4$, $\angle B \cong \angle D$
 $\overline{GB} \cong \overline{HD}$

Prove: $\angle F \cong \angle E$

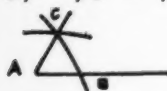


HOMEWORK SOLUTIONS:

- | | | |
|----|--|--|
| 1. | 1) $\overline{CD}$ bisects $\angle ACB$ | 1) Given |
| | 2) $\angle ACD \cong \angle BCD$ (A) | 2) To bisect is to divide into 2 $\cong$ parts |
| | 3) $\overline{CD} \cong \overline{CD}$ (S) | 3) Reflexive princ. of $\cong$ |
| | 4) $\overline{CD} \perp \overline{AB}$ | 4) Given |
| | 5) $\angle CDA$ and $\angle CDB$ are right $\angle$'s | 5) Perpendiculars form right $\angle$'s |
| | 6) $\angle CDA \cong \angle CDB$ (A) | 6) All right $\angle$'s are $\cong$ |
| | 7) $\triangle ACD \cong \triangle BCD$ | 7) ASA $\cong$ ASA |
| | 8) $\overline{AC} \cong \overline{BC}$ | 8) Corr. sides of $\cong \triangle$'s are $\cong$ |
| 2. | 1) $\overline{AD} \cong \overline{CB}$ (S) | 1) Given |
| | 2) $\angle 1 \cong \angle 2$ (A) | 2) Given |

- 3) $\overline{AC} \cong \overline{AC}$ (S)
- 4) $\triangle ADC \cong \triangle CBA$
- 5) $\angle 3 \cong \angle 4$

3a.



- 3b. 1) $\overline{AB} \cong \overline{DE}$ (S)
- 2) $\overline{BC} \cong \overline{EF}$ (S)
- 3) $\overline{AC} \cong \overline{DF}$ (S)
- 4) $\triangle ABC \cong \triangle DEF$
- 5) $\angle CAB \cong \angle FDE$

4. Given: $\overline{AB} \cong \overline{CB}$

$\overline{BD}$ is a median

Prove: $\overline{BD}$ bisects $\angle ABC$

- 1) $\overline{AB} \cong \overline{CB}$ (S)
- 2) $\overline{BD} \cong \overline{BD}$ (S)
- 3) $\overline{BD}$ is a median
- 4) D is a midpoint

5) $\overline{AD} \cong \overline{CD}$ (S)

6) $\triangle ABD \cong \triangle CBD$

7) $\angle 1 \cong \angle 2$

8) $\overline{BD}$ bisects $\angle ABC$

5. 1) $\angle 3 \cong \angle 4$

2) $\angle 1 \cong \angle 4$
 $\angle 2 \cong \angle 3$

3) $\angle 1 \cong \angle 2$ (A)

4) $\overline{GB} \cong \overline{HD}$ (S)

5) $\angle B \cong \angle D$ (A)

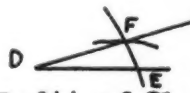
6) $\triangle HDE \cong \triangle GBF$

7) $\angle E \cong \angle F$

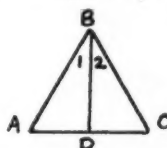
3) Reflexive prop.

4) SAS $\cong$ SAS

5) Corr. $\angle$'s of $\cong \triangle$'s are $\cong$



- 1) Radii of $\cong$ circles are $\cong$
- 2) Radii of $\cong$ circles are $\cong$
- 3) Radii of $\cong$ circles are $\cong$
- 4) SSS $\cong$ SSS
- 5) Corr. $\angle$'s of $\cong \triangle$'s are $\cong$



1) Given

2) Reflexive princ.

3) Given

4) A median is a line segment joining a vertex to the midpoint of the opp. side

5) A midpoint divides a line segment into 2 $\cong$ segments

6) SSS $\cong$ SSS

7) Corr. $\angle$'s of $\cong \triangle$'s are $\cong$

8) To bisect means to divide into 2 $\cong$ parts ($\angle 1 \cong \angle 2$)

1) Given

2) Vertical $\angle$'s are $\cong$

3) Substitution princ.

4) Given

5) Given

6) ASA $\cong$ ASA

7) Corr. $\angle$'s of $\cong \triangle$'s are $\cong$

Lesson 28 - Isosceles Triangle

AIM: What are the properties of an isosceles triangle?

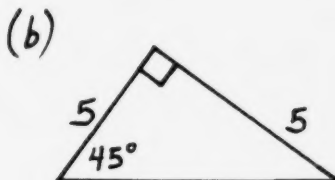
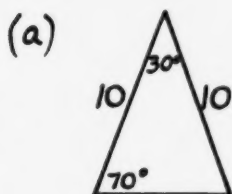
PERFORMANCE OBJECTIVES: The student will be able to...

1. List the properties of an isosceles triangle (reviewed from Course I)
2. Identify the congruent angles, given two congruent sides in simple diagrams and in diagrams with overlapping triangles.
3. Prove the Base Angles Theorem.

VOCABULARY: base angle, vertex angle, base, auxiliary line

MOTIVATION:

Ask the class which of the following triangles can be constructed.



DEVELOPMENT:

1. Guide the pupils to determine that the third angle in (a) measures 80° (from the sum of the measures of the angles of a triangle). This contradicts their intuitive experience with the properties of an isosceles triangle, studied in Course I. They should conclude that triangle (a) is impossible. Triangle (b) is an isosceles right triangle and is possible. From the motivation have the class state, "If two sides of a triangle are congruent, the angles opposite are congruent." Introduce the terms, base, base angle and vertex angle. Rephrase the theorem, "The base angles of an isosceles triangle are congruent."

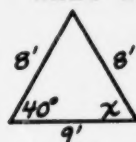
2. (Proof of the theorem) Have the class think about the question, "How do we know the statement is true (valid)?" After a discussion on the need for a proof, show the class a paper isosceles triangle and fold one leg onto the other. Ask whether this would be considered an acceptable proof. (No, in our development we do not have axioms to justify the operation of folding. It is possible to introduce such axioms as is done in some texts.)

"How could we accomplish the same effect as folding the triangle? The line down the middle can be obtained by means of a construction. Which construction might we call for here?"

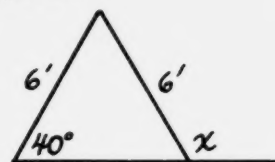
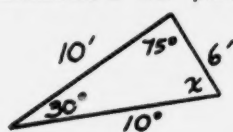
"Since the angle bisector is the easiest, let's see where it leads. If we bisect the vertex angle, what does the bisector create?" Obtain a statement of congruent triangles from the class. Have the students write the steps in a formal proof.

3. Practice identifying congruent base angles.

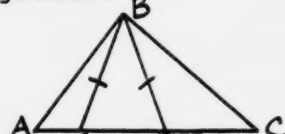
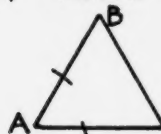
- a) What is the measure of $\angle X$?



($40^\circ, 75^\circ, 140^\circ$)



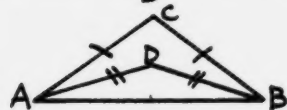
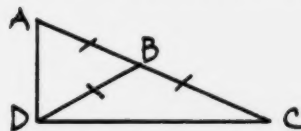
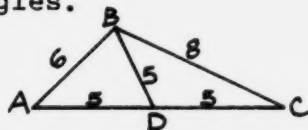
- b) Which angles are congruent?



($\angle B \cong \angle C$, $\angle A \cong \angle A$, $\angle BED \cong \angle BDE$)

4. Do application #1.

5. (More complicated diagrams). Name pairs of congruent angles.



6. Do application #2.

7. (Study proofs informally)

a) If $\overline{AB} \cong \overline{AC}$ and $\overline{EB} \cong \overline{DC}$,

how would you prove $\angle x \cong \angle y$?

(subtraction axiom and base angles theorem)

b) Given: $\overline{AB} \cong \overline{AC}$, which angles

are congruent?

($\angle ABC \cong \angle ACB$)

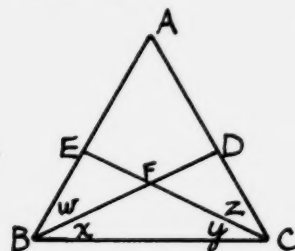
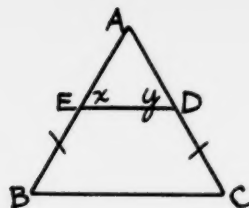
If it is also given that $\overline{FB} \cong \overline{FC}$,

which angles are congruent?

($\angle x \cong \angle y$)

How would you prove $\angle w \cong \angle z$?

(subtraction axiom)



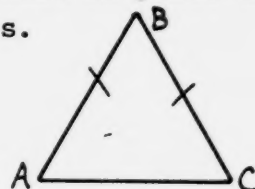
8. Do application #3.

9. (An alternate proof of the theorem)

With a superior class you may want to prove the theorem without drawing auxiliary lines.

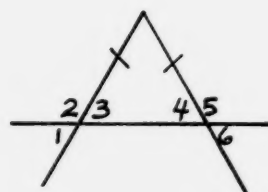
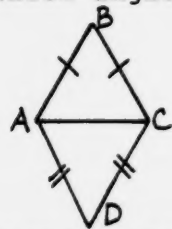
$\triangle ABC \cong \triangle CBA$ by SAS $\cong$ SAS

$\therefore \angle A \cong \angle C$



APPLICATIONS:

1. Which angles are congruent?



($\angle BAC \cong \angle BCA$ and $\angle DAC \cong \angle DCA$;

$\angle 2 \cong \angle 3$ and $\angle 1 \cong \angle 4$;

$\angle 3 \cong \angle 4$, $\angle 2 \cong \angle 5$ and $\angle 1 \cong \angle 6$)

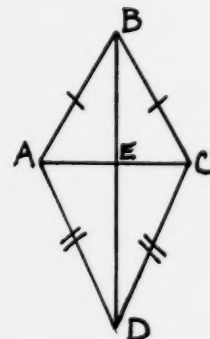
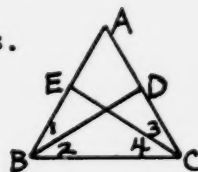
2. Name pairs of congruent angles.

$\overline{AB} \cong \overline{AC}$

$\overline{BD}$ and $\overline{CE}$ are angle bisectors.

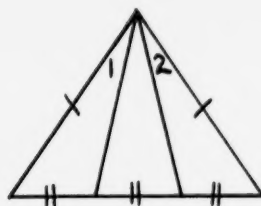
($\angle ABC \cong \angle ACB$, $\angle 1 \cong \angle 2 \cong \angle 3 \cong \angle 4$;

$\angle BAC \cong \angle BCA$ and $\angle DAC \cong \angle DCA$)



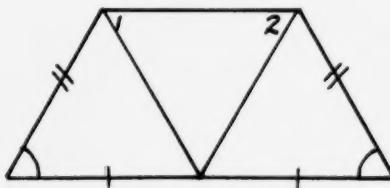
3. Write the plan.

a) Prove: $\angle 1 \cong \angle 2$



(Δ s $\cong$ by SAS)

b) Prove: $\angle 1 \cong \angle 2$

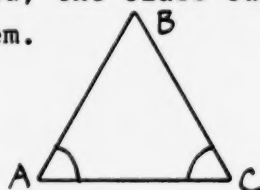


(Δ s $\cong$ by SAS)

4. If the alternate proof is studied, the class can then prove the converse of the theorem.

$\Delta ACB \cong \Delta CAB$ by ASA $\cong$ ASA

$\therefore \overline{AB} \cong \overline{CB}$



SUMMARY:

1. What are the properties of an isosceles triangle?
2. State the base angles theorem.
3. Describe the plan for proving the base angles theorem.

HOMEWORK:

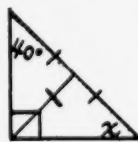
1. Prove the base angle theorem by drawing the median from the vertex.
2. Prove the corollary: The bisector of the vertex angle of an isosceles triangle is perpendicular to the base.

3. Find $m\angle x$

a)

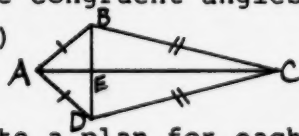


b)

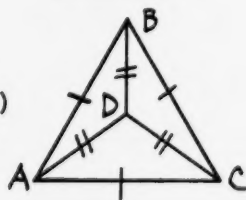


4. Name congruent angles.

a)

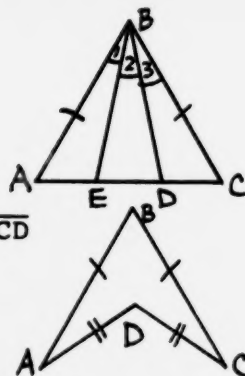


b)



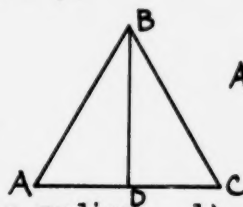
5. Write a plan for each proof.

- a) Given: $\overline{AB} \cong \overline{CB}$
 $\angle 1 \cong \angle 2 \cong \angle 3$
 Prove: $\overline{AE} \cong \overline{CD}$
- b) Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$
 Prove: $\angle A \cong \angle C$



HOMEWORK SOLUTIONS:

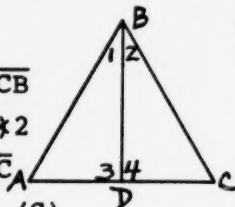
1. Given: $\overline{AB} \cong \overline{CB}$
 Prove: $\angle A \cong \angle C$



- | | |
|---|---|
| 1) Construct a median from B to the midpoint, D, of $\overline{AC}$. | 1) Any line segment may be bisected. There is always a line segment joining two points. |
| 2) $\overline{AB} \cong \overline{CB}$ (S) | 2) Given |
| 3) $\overline{BD} \cong \overline{BD}$ (S) | 3) Reflexive property |
| 4) $\overline{AD} \cong \overline{CD}$ (S) | 4) A bisector divides a line segment into 2 $\cong$ parts. |
| 5) $\triangle ABD \cong \triangle CBD$ | 5) SSS $\cong$ SSS |
| 6) $\angle A = \angle C$ | 6) Corr. $\angle$'s of $\cong \Delta$ s are $\cong$. |

2. Given: $\overline{AB} \cong \overline{CB}$
 $\angle 1 \cong \angle 2$

Prove: $\overline{BD} \perp \overline{AC}$



- | | |
|---|--|
| 1) $\overline{AB} \cong \overline{CB}$ (S) | 1) Given |
| 2) $\angle 1 \cong \angle 2$ (A) | 2) Given |
| 3) $\overline{BD} \cong \overline{BD}$ (S) | 3) Reflexive property of = |
| 4) $\triangle ABD \cong \triangle CBD$ | 4) SAS $\cong$ SAS |
| 5) $\angle 3 \cong \angle 4$ | 5) Corresponding $\angle$'s of = Δ s are = |
| 6) $\angle 3$ suppl. $\angle 4$ | 6) If 2 adjacent $\angle$'s have their non-adjacent sides in a straight line they are supplementary |
| 7) $\angle 3$ and $\angle 4$ are | 7) If 2 $\angle$'s are = and supplementary then they are right $\angle$'s |
| 8) $\frac{\text{rt. } \angle \text{'s}}{\overline{BD} \perp \overline{AC}}$ | 8) If 2 lines meet at right $\angle$'s they are perpendicular. |

3. a) $m\angle x = 20$
 b) $m\angle x = 50$

4. a) $\angle ABD \cong \angle ADB$, $\angle CBD \cong \angle CDB$, $\angle ABC \cong \angle ADC$, $\angle BAC \cong \angle DAC$,
 $\angle BCA \cong \angle DCA$, $m\angle AEB = m\angle AED = 90$
- b) $\angle ABD \cong \angle CBD \cong \angle BCD \cong$ etc.
 $\angle BDC \cong \angle CDA \cong \angle ADB$
 $\angle ABC \cong \angle BCA \cong \angle CAB$
5. a) Prove $\triangle ABE \cong \triangle CBD$ by ASA making corresponding parts
 $\cong$
- b) Draw $\overline{BD}$
 Prove $\triangle ABD \cong \triangle CBD$ by SSS (Alternately, draw $\overline{AC}$ and use subtraction.)

Lesson 29 - The Equilateral Triangle

AIM: What are the properties of an equilateral triangle?

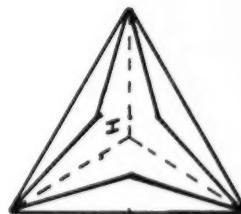
PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove that an equilateral triangle is equiangular.
2. Apply properties of isosceles and equilateral triangles to
 - a) numerical problems.
 - b) proofs.

VOCABULARY: incenter

MOTIVATION:

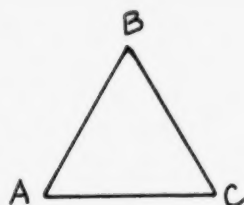
A three-pointed star is based on the equilateral triangle. Use the hint to construct a three-pointed star. Explain your construction.



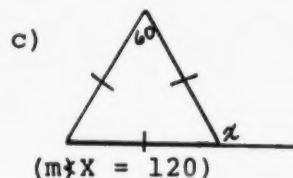
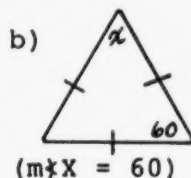
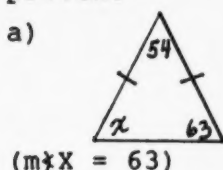
DEVELOPMENT:

1. (Constructions) Discuss the motivation. Have students explain how to find the incenter, I , the point of intersection of the angle bisectors. They would have to bisect the half-angles to find the other vertices of the star. (At present, assume that the 3 angle bisectors are concurrent.) If students have drawn the initial equilateral triangle freehand, it would be best to accept it until the analysis is complete. Then pose the question, "How could we construct an equilateral triangle?" Perform the construction of an equilateral triangle.
2. (Proof) Since the class is involved in bisecting angles it is natural to inquire, "What must be true about the angles of an equilateral triangle?" Have the class prove the corollary.

If $\overline{AB} \cong \overline{BC}$ then $\angle A \cong \angle C$
 If $\overline{AB} \cong \overline{AC}$ then $\angle C \cong \angle B$
 $\therefore \angle A \cong \angle C \cong \angle B$



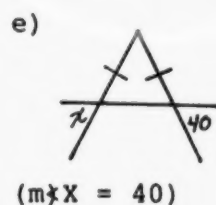
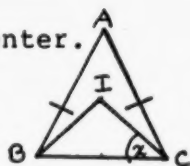
3. (Numericals) In some of the exercises the students may want to introduce the theorem concerning the sum of the measures of the angles of a triangle. Allow them to do so, but point out that the theorem has not yet been proven.



- d) $m\angle ABC = 73$

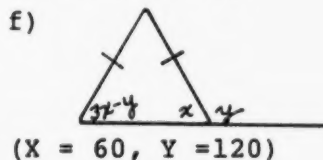
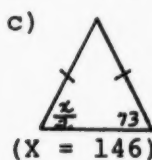
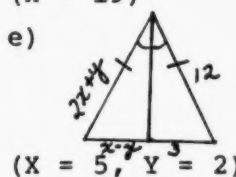
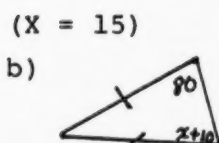
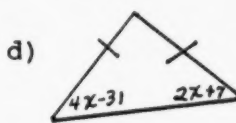
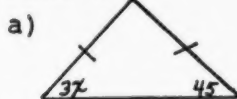
I is the incenter.

$(m\angle X = 36\frac{1}{2})$



4. (Algebra) Present problems in which students have to solve an equation or pairs of equations.

Find x.

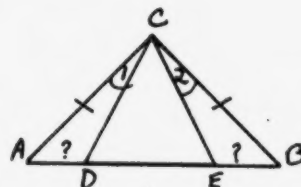


5. (Proofs) In each problem have the class develop a plan before the steps of the proof are written.

a) Given: $\overline{AC} \cong \overline{BC}$

$$\angle 1 \cong \angle 2$$

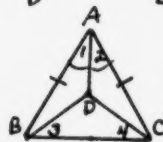
Prove: $\overline{AD} \cong \overline{BE}$
($\cong \Delta$'s by ASA)



b) Given: $\overline{AB} \cong \overline{AC}$

$$\angle 1 \cong \angle 2$$

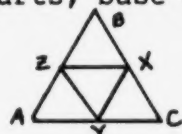
Prove: $\angle 3 \cong \angle 4$
($\cong \Delta$'s by SAS, corr. parts, base angles)



c) Given: ΔABC is equilateral

X, Y, Z are midpoints

Prove: ΔXYZ is equilateral
($\cong \Delta$'s by SAS, using $\frac{1}{2}$'s of = 's are =)



APPLICATIONS:

Make a choice suitable to the ability of the class.

1. (Using the theorem without congruent triangles.)

a) Given: $\overline{AB} \cong \overline{CB}$

I is the incenter.

Prove: $\angle IAC \cong \angle ICA$
(various approaches e.g.
base angles and $\frac{1}{2}$'s
of = 's are =)



b) Given: $\overline{AB} \cong \overline{AC}$

M, N are midpoints

Prove: $\angle x \cong \angle y$
($\frac{1}{2}$'s of = 's
are = and base angles)



2. (Using the theorem with congruent triangles.)

a) Given:

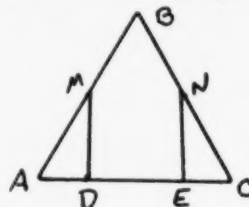
$$\overline{AB} \cong \overline{CB}, \overline{AD} \cong \overline{CE}$$

M, N are midpoints.

Prove:

$$\overline{MD} \cong \overline{NE}$$

(SAS)



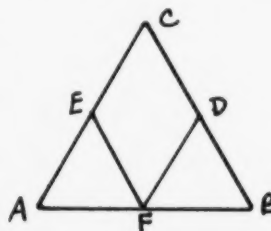
b) Given:

ΔABC is equilateral

E, F, D are midpoints

Prove:

EFDC is a rhombus
($\cong \Delta$'s by SAS)



3. (Drawing auxiliary lines.)

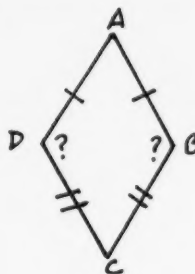
a) Given:

$$\overline{AB} \cong \overline{AD}, \overline{CB} \cong \overline{CD}$$

Prove:

$$\angle B \cong \angle D$$

(2 ways: Draw $\overline{AC}$ or $\overline{BD}$.)
($\cong \Delta$'s by SSS or base angles)



b) Given:

$$\overline{AB} \cong \overline{AC}, \overline{DC} \cong \overline{DB}$$

Prove:

$$\angle C \cong \angle B$$

(2 ways)
(SSS or base angles
with subtraction axiom)



4. (Verbals) Prove: The incenter of an equilateral triangle is equidistant from the vertices. ($\cong \Delta$'s by ASA)

5. (Uses in the world) A

carpenter's level consists

of an isosceles triangle

with the midpoint of the

base marked M. The base is

placed on a beam. If a plumb line dropped from the

vertex passes through M, the beam is horizontal. Why

does it work?

($\cong \Delta$'s by SAS, right $\angle$'s at M, plumb line is
vertical, $\therefore$ beam is horizontal)



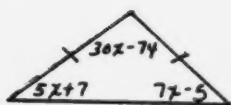
SUMMARY:

1. What is the definition of an equilateral triangle?
2. Name an additional property of the equilateral triangle.
3. An angle bisector in an equilateral triangle is an axis of symmetry. How many axes of symmetry does an equilateral triangle have? What does the axis of symmetry do?

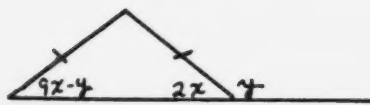
HOMEWORK:

1. Find x .

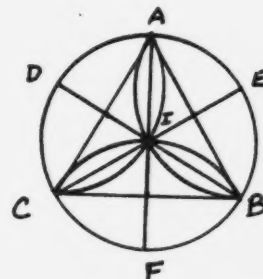
a)



b)

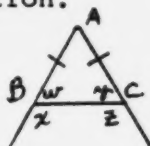


2. The three-leafed star is based on the equilateral triangle and its incenter. Use the hint to the right to construct a three-leafed star. Explain your construction.



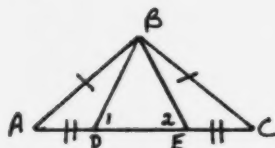
3. Given: $\overline{AB} \cong \overline{AC}$

Prove: $\angle x \cong \angle z$



4. Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CE}$

Prove: $\angle 1 \cong \angle 2$



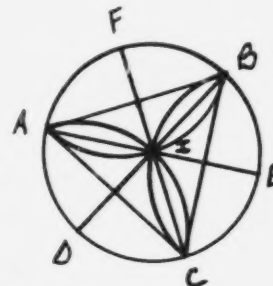
5. Prove: If a median of a triangle is also an altitude, then the triangle is isosceles.

HOMEWORK SOLUTIONS:

1. a) $5x + 7 = 7x - 5$
 $12 = 2x$
 $6 = x$

b) $2x + y = 180$
 $9x - y = 2x$
 $y = 7x$
 $9x = 180$
 $x = 20$

2. Bisect each angle of the equilateral triangle to find I. Use I as the center of a circle which circumscribes the triangle. Extend the angle bisectors to points D, E and F. Use D as the center of a circle with radius DI. Draw arc AIC. Use points E and F in a similar way.



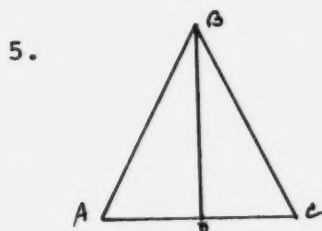
3. 1) $\overline{AB} \cong \overline{AC}$
 2) $\angle w \cong \angle y$

- 1) Given
 2) The base $\angle$'s of an isosceles $\triangle$ are =

- 3) $\angle x$ suppl. of $\angle w$
 $\angle z$ suppl. of $\angle y$

4) $\angle x \cong \angle z$ 4)

4. 1) $\overline{AB} \cong \overline{CB}$ (S)
 2) $\angle A \cong \angle C$ (A)
 3) $\overline{AD} \cong \overline{CE}$ (S)
 4) $\triangle ABD \cong \triangle CBE$
 5) $\overline{BD} \cong \overline{BE}$
 6) $\angle 1 \cong \angle 2$



- 1) $\overline{BD}$ is a median
 2) D is the midpoint of $\overline{AC}$
 3) $\overline{AD} \cong \overline{CD}$ (S)
 4) $\overline{BD}$ is an altitude
 5) $\overline{BD} \perp \overline{AC}$
 6) $\angle BDA$ is a rt. $\angle$
 $\angle BDC$ is a rt. $\angle$
 7) $\angle BDA \cong \angle BDC$ (A)
 8) $\overline{BD} \cong \overline{BD}$ (S)
 9) $\triangle ABD \cong \triangle CBD$
 10) $\overline{AB} \cong \overline{CB}$
 11) $\triangle ABC$ is isos.

- 3) If 2 adj. $\angle$'s have their ext. sides in a straight line then they are supplementary

- 4) Supplements of $\cong \angle$'s are $\cong$

- 1) Given
 2) Base $\angle$'s of an isosceles $\triangle$ are $\cong$
 3) Given
 4) SAS $\cong$ SAS
 5) Corr. sides of $\cong \triangle$'s are $\cong$
 6) Base $\angle$'s of an isosceles $\triangle$ are $\cong$

Given: $\overline{BD}$ is a median

$\overline{BD}$ is a altitude

Prove: $\triangle ABC$ is isosceles

- 1) Given
 2) A median is a line segment that joins a vertex to the midpoint of the opposite side.
 3) A midpoint divides a line segment into 2 $\cong$ parts
 4) Given
 5) An alt. is a line segment from a vertex $\perp$ to the opposite side terminating there.
 6) Perpendiculars form right $\angle$'s
 7) All right $\angle$'s are $\cong$
 8) Reflex. prop. of $\cong$
 9) SAS $\cong$ SAS
 10) Corr. sides of $\cong \triangle$'s are $\cong$
 11) A $\triangle$ with 2 $\cong$ sides is isosceles.

Lesson 30 - Congruent Triangles

AIM: How do we prove overlapping triangles are congruent?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Plan a proof involving overlapping triangles by
 - a) outlining the triangles in color.
 - b) separating the triangles.
 - c) correctly using previous techniques to decide which method of congruence is appropriate.
2. Write out proofs involving overlapping triangles.

VOCABULARY: overlapping triangles, separating a pair of triangles

MOTIVATION:

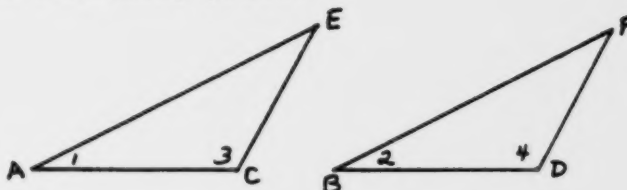
1. Have the class do the following proof:

Given: $\angle 1 \cong \angle 2$

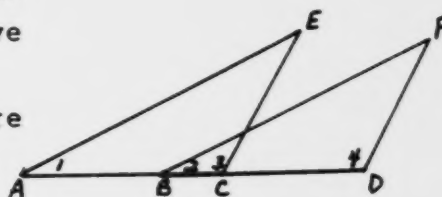
$\angle 3 \cong \angle 4$

$\overline{AC} \cong \overline{BD}$

Prove: $\overline{CE} \cong \overline{DF}$



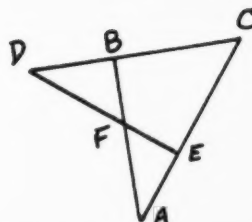
2. Using two transparencies (one for each triangle) and an overhead projector, move the triangles into the following position (be sure that each triangle was originally outlined in a different color):



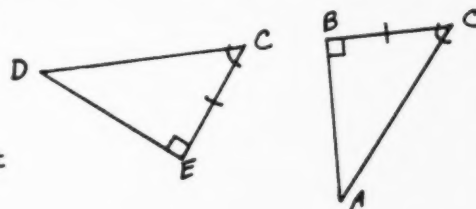
3. Point out that had the original problem been given with this diagram instead, it would have been more difficult since the triangles "overlap".
4. In the next few lessons it will be important to be able to prove overlapping triangles congruent.

DEVELOPMENT:

- Given: $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$, $\overleftrightarrow{DE} \perp \overleftrightarrow{CA}$
 $\overline{BC} \cong \overline{EC}$
 Prove: $\overline{CD} \cong \overline{CA}$



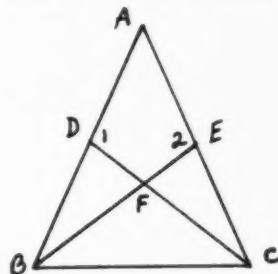
- Recall from earlier: To prove two segments congruent, we could prove that they are corresponding parts of congruent triangles. Since there are several triangles in this diagram, we must take special care that we are using the correct ones. Which triangles should we try to prove congruent for this problem? ($\triangle CAB = \triangle CDE$)
- In order to simplify this problem, redraw the diagram, SEPARATING the triangles, using different colors.



- Mark the diagrams as above, write out the plan and have the class write out the proof at their seats.

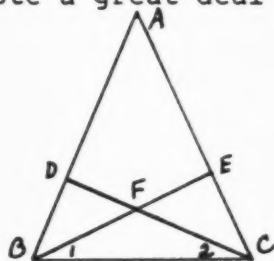
APPLICATIONS:

- Given: $\overline{AB} \cong \overline{AC}$;
 E is the midpoint of $\overline{AC}$;
 D is the midpoint of $\overline{AB}$
 Prove: $\angle 1 \cong \angle 2$



(Note that it is essential to choose the pair of triangles you want to work with carefully since a wrong choice will a) waste a great deal of time and b) not do the job!)

- Given: $\overline{DB} \cong \overline{EC}$
 $\angle ABC \cong \angle ACB$
 Prove: $\angle 1 \cong \angle 2$



These examples show two possible ways of using the reflexive property with overlapping triangles.

SUMMARY: When a congruence proof involves a complicated diagram:

1. First find two triangles containing the corresponding parts that, hopefully, are congruent.
2. If the triangles overlap, use two different colors to outline the triangles.
3. You may wish to "separate" the triangles to help do the analysis.

HOMEWORK:

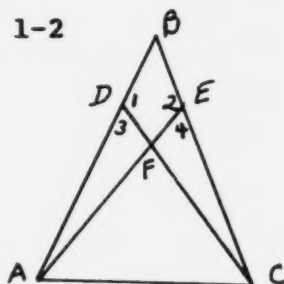
1. Given: $\overline{DA} \cong \overline{EC}$
 $\angle DAC \cong \angle ECA$
 Prove: $\angle 3 = \angle 4$

2. Given: $\overline{DB} \cong \overline{EB}$
 $\overline{BC} \cong \overline{BA}$
 Prove: $\angle 1 \cong \angle 2$

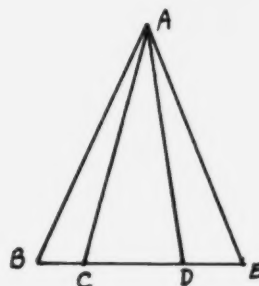
3. Name every triangle that contains $\angle B$.
4. Name every triangle that contains $\overline{AB}$.
5. Name every triangle that contains $\overline{AC}$.
6. Name every triangle that contains $\overline{BC}$.

7. Name every triangle that contains $\overline{BD}$.
8. Name every triangle that contains $\overline{AC}$.
9. Name every triangle that contains $\angle ABC$.
10. Name every triangle that contains $\overline{BC}$.

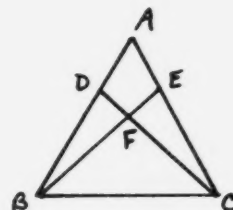
ex. 1-2



ex. 3-6



ex. 7-10



HOMEWORK SOLUTIONS:

1. Prove $\triangle DAC \cong \triangle ECA$ by SAS = SAS.
2. Prove $\triangle BDC \cong \triangle BEA$ by SAS = SAS.
3. $\triangle ABC$, $\triangle ABD$, $\triangle ABE$
4. $\triangle ABC$, $\triangle ABD$, $\triangle ABE$
5. $\triangle ABC$, $\triangle ACD$, $\triangle ACE$
6. $\triangle ABC$
7. $\triangle BDF$, $\triangle BDC$
8. $\triangle ABC$, $\triangle DAC$
9. $\triangle ABC$, $\triangle DBC$
10. $\triangle ABC$, $\triangle DBC$, $\triangle EBC$, $\triangle FBC$

Lesson 31 - Congruent Triangles

AIM: How do we do a more difficult overlapping triangles proof?

PERFORMANCE OBJECTIVES: The student will be able to...

Plan and prove a more difficult overlapping triangles proof.

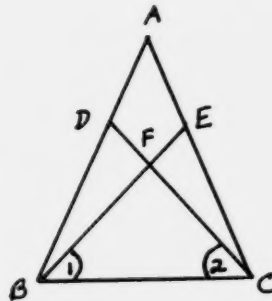
VOCABULARY: overlapping triangles

MOTIVATION:

Name pairs of triangles we may desire to prove congruent in order to do this proof.

Given: $\overline{AB} \cong \overline{AC}$, $\angle 1 \cong \angle 2$

Prove: $\overline{DB} \cong \overline{EC}$



DEVELOPMENT:

1. The above example stresses that we must focus on the "PROVE" part of a proof in order to decide upon the triangles to use.
2. Discuss the fact that we have basically TWO possibilities here:
 - a) Prove $\triangle DFB \cong \triangle EFC$ or
 - b) $\triangle DCB \cong \triangle ECB$(A very bright group may even suggest how we can use triangles ABE and ACD.)
3. Elicit that as a result of the GIVEN, it would be too difficult to prove $\triangle DFB \cong \triangle EFC$.
4. Elicit the plan using $\triangle DCB \cong \triangle ECB$ by ASA = ASA.
5. Note that often our first choice of triangles must be rejected and we must try another pair.
6. Have the class write out the proof to the previous problem at their seats. One possibility is:

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$	1. Given.
2. $\angle DBC \cong \angle ECB$ ($A \cong A$)	2. If two sides of a triangle are congruent then the angles opposite them are also congruent.
3. $\overline{BC} \cong \overline{BC}$ ($S \cong S$)	3. Reflexive Property.
4. $\angle 1 \cong \angle 2$ ($A \cong A$)	4. Given.
5. $\triangle DCB \cong \triangle ECB$	5. Two triangles are congruent if two angles and the included side of one are congruent to two angles and the included side of the other.
6. $\overline{DB} \cong \overline{EC}$	6. Corresponding sides of congruent triangles are congruent.

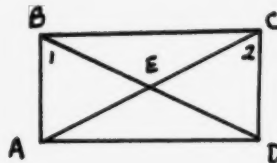
APPLICATIONS:

For each problem:

- List all possible triangle pairs we could use to do the proof.
- Tell which pair we could use most successfully based on the given.
- Outline a plan.

1. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$,
 $\overleftrightarrow{DC} \perp \overleftrightarrow{BC}$,
 $\angle 1 \cong \angle 2$

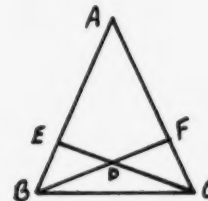
Prove: $\overline{AB} \cong \overline{DC}$



Given (for 2-3): $\overline{BE} \cong \overline{CF}$,
 $\overline{EA} \cong \overline{FA}$

- Prove: $\overline{BF} \cong \overline{CE}$
- Prove: $\angle FBC \cong \angle ECB$

(Note: In example 2, either pair of triangles can be used successfully.)



SUMMARY:

When triangles overlap, you must be sure to:

- Focus on the parts mentioned in the "PROVE" section.
- Find pairs of triangles that contain the parts you WANT to be congruent.
- Try to prove a pair of triangles congruent. If that fails, try another pair of triangles containing the parts you WANT congruent.

HOMEWORK:

For each problem:

- List all possible triangle pairs we could use to do the proof.
- Tell which pair we could most successfully use based on the given.
- Outline the plan.

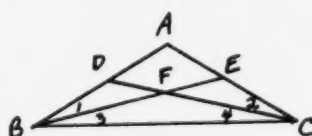
For 1-2:

Given: $\overline{AB} \cong \overline{AC}$,
 $\overline{DB} \cong \overline{EC}$

1. Prove: $\angle 1 \cong \angle 2$

2. Prove: $\angle 3 \cong \angle 4$

ex. 1-4



For 3-4:

Given: $\overrightarrow{BE}$ bisects $\angle ABC$, $\overrightarrow{CD}$ bisects $\angle ACB$, $\overline{AB} = \overline{AC}$

3. Prove: $\overline{AD} \cong \overline{AE}$

4. Prove: $\overline{DB} \cong \overline{EC}$

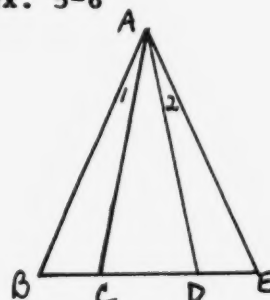
5. Given: $\angle 1 \cong \angle 2$, $\overline{AC} \cong \overline{AD}$

Prove: $\overline{DB} \cong \overline{CE}$

6. Given: $\angle 1 \cong \angle 2$, $\overline{AC} \cong \overline{AD}$

Prove: $\overline{AB} \cong \overline{AE}$

ex. 5-6



HOMEWORK SOLUTIONS:

1. a) $\triangle BDF \cong \triangle CEF$; $\triangle BAE \cong \triangle CAD$
 b) $\triangle BAE \cong \triangle CAD$
 c) $\triangle BAE \cong \triangle CAD$ by SAS $\cong$ SAS
2. a, b) $\triangle BEC \cong \triangle CDB$
 c) use SAS $\cong$ SAS
3. a, b) $\triangle ADC \cong \triangle AEB$
 c) use ASA $\cong$ ASA
4. a, b) $\triangle BDC \cong \triangle CEB$
 c) use ASA $\cong$ ASA
5. a, b) $\triangle BAD \cong \triangle EAC$
 c) use ASA $\cong$ ASA
6. a) $\triangle ABC \cong \triangle AED$, $\triangle ABD \cong \triangle AEC$
 b) $\triangle ABD \cong \triangle AEC$
 c) use ASA $\cong$ ASA

Lesson 32 - Congruent Triangles

AIM: How do we use TWO pairs of congruent triangles in a proof?
(Part 1)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Plan a proof that two segments (or angles) are congruent by:
 - a) proving two triangles congruent, then
 - b) correctly using corresponding parts of congruent triangles together with other information to prove two other triangles congruent.
 - c) using corresponding parts of congruent triangles are congruent.
2. Write out such a proof.

VOCABULARY: double congruence

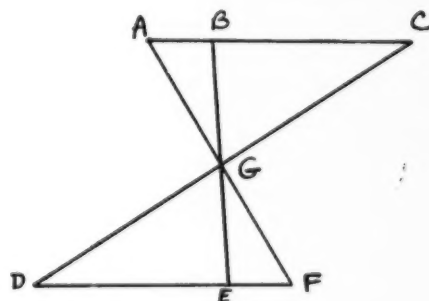
MOTIVATION:

1.
 - a) What triangles should we prove congruent in order to do this problem?
 - b) Write out a plan for the proof.

Given: B lies on $\overline{AC}$ and E lies on $\overline{FD}$.

$\overline{AF}$ and $\overline{BE}$ bisect each other.

Prove: $\overline{AC} \cong \overline{FD}$



2. Elicit that WE WANT $\triangle ACG \cong \triangle FDG$. However, the given information allows us to prove $\triangle ABG \cong \triangle FEG$ and not any other pair of triangles congruent directly.
3. We can use the fact that $\triangle ABG \cong \triangle FEG$ to HELP PROVE that $\triangle ACG \cong \triangle FDG$ as well. (Elicit that $\angle A \cong \angle F$ results from $\triangle ABG \cong \triangle FEG$ and can be used to prove the other pair of triangles are congruent. Also use two colors to outline

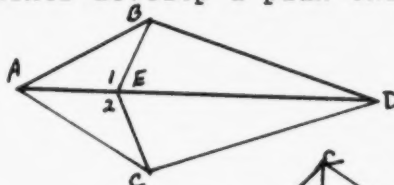
which triangles are congruent.) Often, in order to prove a pair of triangles are congruent, we must first prove another pair are congruent.

DEVELOPMENT:

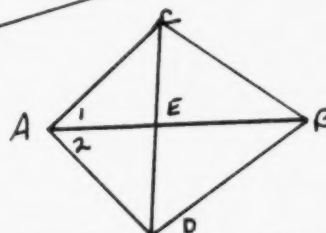
1. The plan in the problem becomes:
 - a) First prove $\triangle ABG \cong \triangle FEG$ by SAS $\cong$ SAS.
 - b) Use $\angle A \cong \angle F$ as a result of a).
 - c) Now prove $\triangle ACG \cong \triangle FDG$ by ASA $\cong$ ASA.
 - d) $\overline{AC} \cong \overline{FD}$ because they are corresponding sides of congruent triangles.
2. Have the class write out the proof.
3. Ask, "Why were we forced to use TWO pairs of congruent triangles here? How do you know whether to prove one or two pairs of triangles congruent?"
 Discuss the fact that we use a "double congruence" proof only as a "last resort".

APPLICATIONS: Have the students develop a plan only.

1. Given: $\overleftrightarrow{AED}$,
 $\overline{AB} \cong \overline{AC}$, $\overline{BD} \cong \overline{CD}$
 Prove: $\angle 1 \cong \angle 2$



2. Given: $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$
 $\angle 1 \cong \angle 2$
 Prove: $\overline{BC} \cong \overline{BD}$

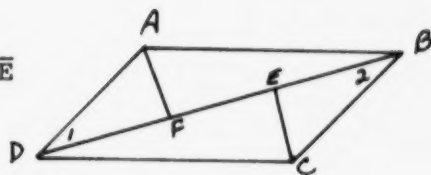


SUMMARY: A "Double Congruence" proof involves four steps.

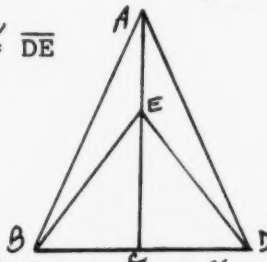
1. Proving a pair of triangles congruent.
2. Choosing a pair of corresponding parts of congruent triangles with the given information to
3. Prove another pair of triangles congruent.
4. Using "corresponding parts of congruent triangles are congruent".

HOMEWORK:

1. Given: $\overline{AB} \cong \overline{CD}$, $\overline{BE} \cong \overline{DF}$, $\overline{AF} \cong \overline{CE}$
 Prove: $\angle 1 \cong \angle 2$



2. Given: $\overline{AB} \cong \overline{AD}$, $\overline{BE} \cong \overline{DE}$
 Prove: $\overline{BC} \cong \overline{DC}$



HOMEWORK SOLUTIONS:

1. First prove $\triangle ABF \cong \triangle CDE$ by SSS $\cong$ SSS. Then, either
 a) $\triangle AFD \cong \triangle CEB$ by SAS $\cong$ SAS or
 b) $\triangle DAB \cong \triangle BCD$ by SAS $\cong$ SAS.
2. First prove $\triangle ABE \cong \triangle ADE$ by SSS $\cong$ SSS. Then, either
 a) $\triangle BEC \cong \triangle DEC$ by SAS $\cong$ SAS or
 b) $\triangle ABC \cong \triangle ADC$ by SAS $\cong$ SAS.

Lesson 33 - Congruent Triangles

AIM: How do we use TWO pairs of congruent triangles in a proof?
(Part 2)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Plan a proof that two segments (or angles) are congruent by:
 - a) proving two triangles congruent, then
 - b) correctly using corresponding parts of congruent triangles together with other information to prove two other triangles congruent.
 - c) using corresponding parts of congruent triangles are congruent.
2. Write out such a proof.

VOCABULARY: double congruence

MOTIVATION:

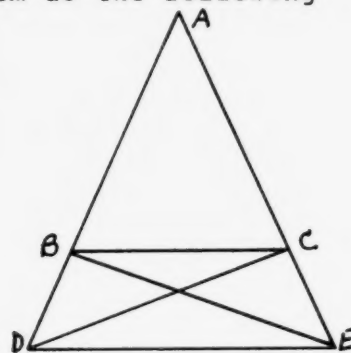
Without using the "base angles" theorem do the following proof.

Given: Isosceles $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$
 $\overline{AB}$ is extended to D and
 $\overline{AC}$ is extended to E so that
 $\overline{BD} \cong \overline{CE}$

Prove: a) $\angle BDE \cong \angle CED$
b) $\angle DBC \cong \angle ECB$
c) $\angle ABC \cong \angle ACB$

(Note to teacher: First prove $\triangle ABE \cong \triangle ACD$ by SAS, then $\triangle DBC \cong \triangle ECB$ by SAS $\cong$ SAS).

In the above problem, using double congruence, we have proven something very interesting. What important result of geometry does this example prove?

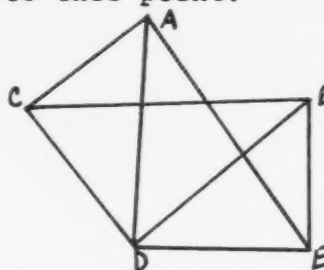


DEVELOPMENT:

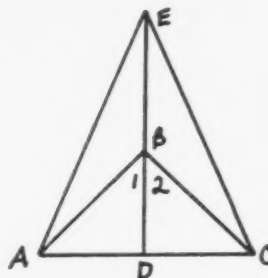
1. The above problem is an alternate proof to the "base angles" theorem: "If two sides of a triangle are congruent, then the angles opposite these sides are also congruent."
2. Euclid proved the "base angles" theorem this way.
However, since most people consider this proof to be more difficult, we proved it earlier, using a different approach. (Ask the class to recall the previous approach.) Euclid's proof could not use angle-bisector since construction was considered later!
A historical note: Because this proof was considered so difficult, the "base angles" theorem had two Latin names: a) "Pons Asinorum" or "Bridge of Fools", since the figure resembles the truss of a bridge and b) "Elefuga" or "Flight of the Miserable Ones" since so many students fled from geometry when they got to this point!

APPLICATIONS:

1. Given: $\overline{EB} \cong \overline{CA}$, $\overline{DE} \cong \overline{DC}$,
 $\overline{DB} \cong \overline{DA}$
Prove: $\overline{CB} \cong \overline{EA}$



2. (If time)
Given: $\overleftrightarrow{ED} \perp \overleftrightarrow{AC}$,
 $\angle 1 \cong \angle 2$
Prove: $\triangle ACE$ is an isosceles triangle.



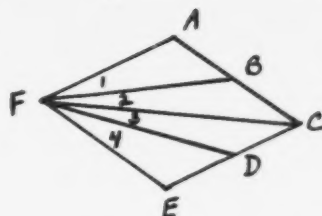
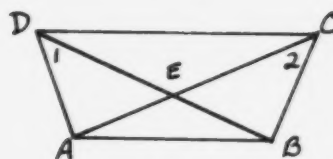
SUMMARY:

When two segments or angles must be proved congruent, locate two triangles that contain those parts and try to prove the triangles congruent. You may need to prove TWO OTHER triangles congruent as well in order to show more parts of the original pair of triangles congruent.

HOMEWORK:

1. Name all triangles in the diagram.
2. Given: $\overline{AD} \cong \overline{BC}$, $\overline{BD} \cong \overline{AC}$
 $\angle 1 \cong \angle 2$
 Prove: $\overline{ED} \cong \overline{EC}$
3. Given: $\overline{AF} \cong \overline{EF}$, $\overline{AC} \cong \overline{EC}$
 $\overline{BC} \cong \overline{DC}$
 Prove: $\angle 2 \cong \angle 3$

ex. 1-2



HOMEWORK SOLUTIONS:

1. $\triangle ABE$, $\triangle ADB$, $\triangle ACB$, $\triangle ADE$, $\triangle ACD$, $\triangle BCE$, $\triangle BCD$, $\triangle CED$
2. First prove $\triangle ADB \cong \triangle BCA$ by SAS $\cong$ SAS.
 Then prove $\triangle ADE \cong \triangle BCE$ by ASA $\cong$ ASA.
3. First prove $\triangle FAC \cong \triangle FEC$ by SSS $\cong$ SSS.
 Then prove $\triangle FBC \cong \triangle FDC$ by SAS $\cong$ SAS.

Lesson 34A - Parallelism

AIM: How do we prove that lines are perpendicular? (Part 1)

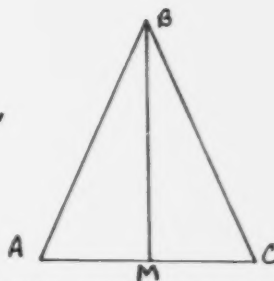
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define perpendicularity, equidistant and use the symbol " $\perp$ ".
2. State the two theorems:
 - a) "If two lines intersect to form congruent adjacent angles, then the lines are perpendicular."
 - b) "If two points are each equidistant from the ends of a line segment, the points determine the perpendicular bisector of the line segment."
3. Apply these two theorems in numerical and algebraic situations.
4. Apply these two theorems in original proofs that lines are perpendicular.

VOCABULARY: median, equidistant

MOTIVATION:

1. Given: In isosceles $\triangle ABC$,
 $\overline{AB} \cong \overline{CB}$ and median $\overline{BM}$ is
drawn to $\overline{AC}$.
Prove: $\overleftrightarrow{BM} \perp \overleftrightarrow{AC}$



2. In the above problem, students must prove that $\angle BMA$ and $\angle BMC$ are right angles. $\underline{x|x}$ Show that $\overline{BM}$ and $\overline{AC}$ intersect to form congruent adjacent angles; $x + x = 180$, so $x = 90$. Elicit the theorem, "If two lines intersect to form congruent adjacent angles, the lines are perpendicular."
3. The result in the above problem may be stated as a theorem, "The median to the base of an isosceles triangle is also the altitude."

4. In this lesson we are interested in proving two lines are perpendicular.

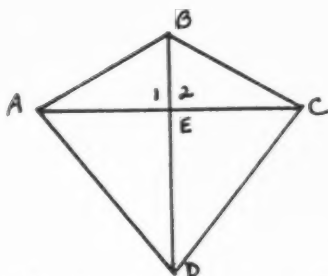
DEVELOPMENT:

Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$

Prove: a) $\angle 1 \cong \angle 2$

b) $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$

c) $\overline{AE} \cong \overline{CE}$



1. In the above problem, students must use a double congruence. First prove $\triangle ABD \cong \triangle CBD$ by SSS $\cong$ SSS. Then prove $\triangle ABE \cong \triangle CBE$ by SAS $\cong$ SAS. The needed results come directly.
2. Very often we have the above situation in a problem. As a result of the above proof, we can avoid going through a double congruence proof to make a conclusion.
3. Note: Point B is the same distance from point C that it is from point A.

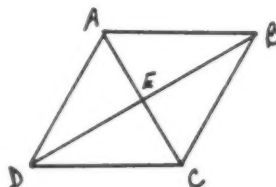
We say that point B is "EQUIDISTANT" from A and C.

Ask: As a result of the given, what other point is EQUIDISTANT from two points? (D is equidistant from A and C). We can state the results of the problem as:

Theorem: If two points are each equidistant from the ends of a line segment, the points determine the perpendicular bisector of the segment.

APPLICATIONS:

1. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E and $m\angle AED = 3x$ and $m\angle DEB = 4x - 30$, a) find x and b) prove $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.
2. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E and $m\angle AEC = 9x$ and $m\angle DEB = 10x - 10$, prove $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.
3. Given: $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$
Prove: $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$



SUMMARY: We now have three ways of proving that lines are perpendicular.

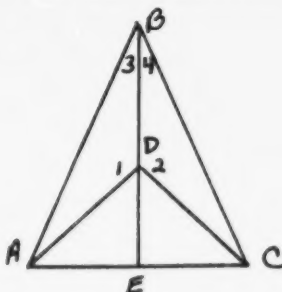
1. If two lines intersect to form right angles, then they are perpendicular.
2. If two lines intersect to form congruent adjacent angles, then they are perpendicular.
3. If two points are each equidistant from the endpoints of a segment, the two points lie on the perpendicular bisector of the segment.

HOMEWORK:

1. Given: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$

Prove: $\overleftrightarrow{BE}$ is the

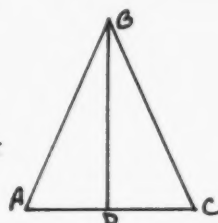
perpendicular bisector of $\overline{AC}$.



2. Given: $\overline{AB} \cong \overline{CB}$

$\overline{BD}$ bisects $\angle ABC$.

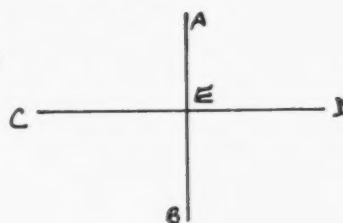
Prove: $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.



3. Given: $m\angle AED = 4x + 2$

$$m\angle AEC = 5x - 20$$

Prove: $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$



HOMEWORK SOLUTIONS:

1. Prove $\triangle BAD \cong \triangle BCD$ by ASA $\cong$ ASA. Since both B and D are equidistant from A and C, $\overleftrightarrow{BE}$ is the perpendicular bisector of $\overline{AC}$.
2. Since $\triangle BAD \cong \triangle BCD$, $\angle BDA \cong \angle BDC$. Since $\overline{BD}$ and $\overline{AC}$ form congruent adjacent angles, $\overline{BD} \perp \overline{AC}$.
3. $x = 22$, so $\angle AEC$ and $\angle AED$ are right angles. If two lines form right angles, then they are perpendicular.

Lesson 34B - Parallelism

AIM: How do we prove that lines are perpendicular? (Part 2)

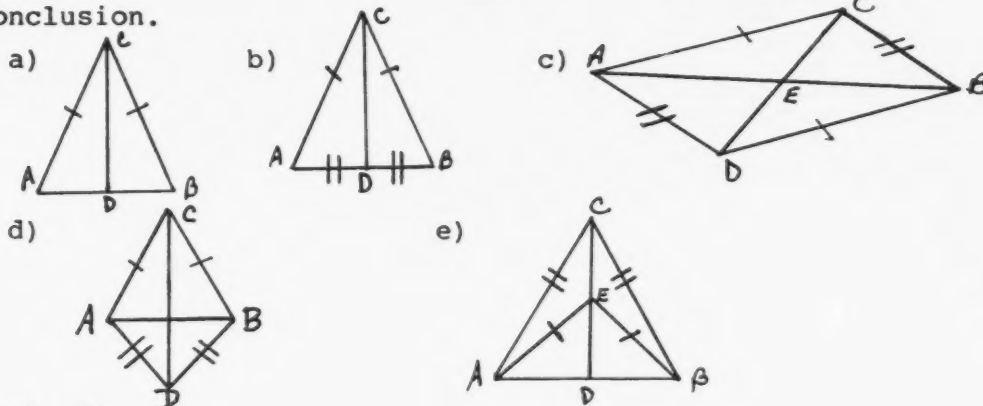
PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the two theorems from the previous lesson in original proofs that lines are perpendicular.
2. Construct a perpendicular to a given line at a point on or outside the line.
3. Justify the constructions above with the appropriate theorem.

VOCABULARY: equidistant, median, perpendicular bisector

MOTIVATION:

1. From the information given in each case by the diagram, determine whether or not $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ and justify each conclusion.



- a) No.
- b) Yes, since C and D are equidistant from A and B.
- c) No.
- d) Yes, since C and D are equidistant from A and B.
- e) Yes, since C and E are equidistant from A and B.

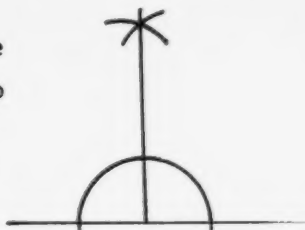
LEAVE THESE FIVE DIAGRAMS ON THE SIDE BOARD.

2. Earlier this term we did some constructions (Lesson 15). Some of the above examples justify our techniques.

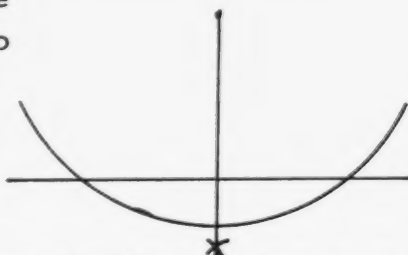
DEVELOPMENT:

1. On the board, recall these two constructions. Elicit the METHODS from the class. (Hopefully, recent homework reviewed it.)

- a) Construct a line perpendicular to a given line through a point on the line.

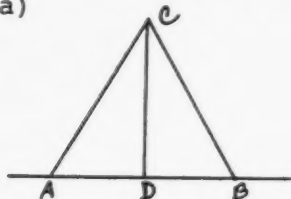


- b) Construct a line perpendicular to a given line through a point outside the line.

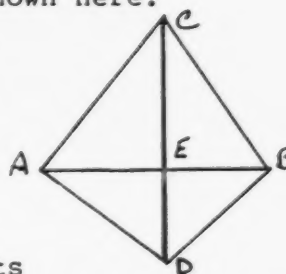


2. Redraw the two diagrams WITHOUT the construction lines, label and connect points as shown here.

a)



b)



3. Point out that certain segments in each diagram are congruent. Why? Show in diagram a) that $\overline{AC}$ and $\overline{BC}$ are both RADII of the same circle (see original diagram a). Elicit that ALL radii of the same circle are congruent. Radii of congruent circles are also congruent. Thus, $\overline{AC} \cong \overline{BC}$. (You may have to complete a circle from each endpoint of the arc to emphasize this.) We now have the same situation as in example b) from the beginning of the lesson. In this example, we showed that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$. (Recall reason.) This explains WHY our construction works.

4. Can you mark the congruent segments in diagram b)?
($\overline{AC} \cong \overline{BC}$, $\overline{AD} \cong \overline{BD}$)

Ask: Have we seen this situation earlier in today's

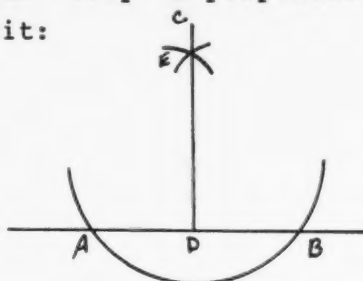
lesson? Example d) earlier today is the same diagram.
We already showed that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

APPLICATIONS:

1. Example e) (earlier) justifies an ALTERNATIVE way to do the second construction ("Dropping a Perpendicular").

Ask: "Can you use example e) to figure out another way we can "drop" a perpendicular from a point to a line?

Elicit:



By drawing arcs from A and B "ABOVE" the line, we get the same situation as in example e). Show this by connecting $\overline{AE}$, $\overline{AC}$, $\overline{BE}$ and $\overline{BC}$ afterwards.

2. Can you JUSTIFY our construction for the perpendicular bisector of a segment? Recall the construction (diagram 1).

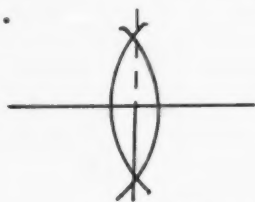
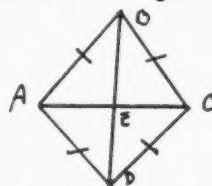


diagram 1

Elicit diagram 2.



- a) This diagram was in last night's homework which showed that $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$.
- b) Why must $\overline{AE} \cong \overline{CE}$? ($\overleftrightarrow{BD}$ must be $\perp$ bisector of $\overline{AC}$ - from homework.)

SUMMARY:

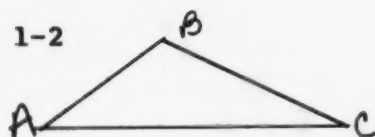
All constructions in geometry have justifications. Often they rely on:

- a) All radii of a circle are congruent.
- b) Two points equidistant from the endpoints of a segment lie on the perpendicular bisector of the segment. And/or
- c) Congruent triangles (even "double congruence").

HOMEWORK:

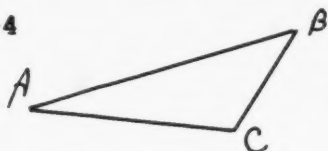
1. a) Copy $\triangle ABC$.
Construct altitude $\overline{BD}$ to $\overline{AC}$.

ex. 1-2



- b) Justify your construction with a theorem.
2. a) Copy $\triangle ABC$. Construct the median $\overline{BE}$ to $\overline{AC}$.
- b) Justify your construction with a theorem.

ex. 3-4



Redo ex. 1) and 2) for the $\triangle ABC$ shown here.

HOMEWORK SOLUTIONS:

1. a) Drop a $\perp$ to $\overline{AC}$.
- b) Two points equidistant from the endpoints of a segment will determine its perpendicular bisector.
2. a) Same construction as 2) in the application section of this lesson.
- b) Two points equidistant from A and C will lie on the bisector of $\overline{AC}$; then connect the midpoint of $\overline{AC}$ to B.
3. Same as 1) above BUT you must first extend $\overline{AC}$ in order that an arc from B intersects a segment in two points.
4. Same as 2) above.

Lesson 35 - Parallelism

AIM: What are parallel lines?

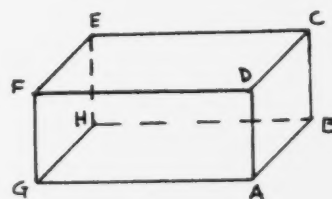
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define parallel lines and transversal.
2. Use the symbol $//$ to indicate parallelism.
3. State the reflexive, symmetric, and transitive postulates for parallelism.
4. State the parallel postulate.
5. Identify pairs of alternate interior angles, corresponding angles and interior angles on the same side of the transversal.
6. Postulate that "If a pair of lines are cut by a transversal forming congruent alternate interior angles, the lines are parallel."

VOCABULARY: alternate interior angles, corresponding angles, interior angles on the same side of the transversal, skew lines

MOTIVATION:

1. This diagram represents a classroom (hopefully the one in which this lesson is being given). The plane formed by ABCD represents the front board. Which plane represents the ceiling? ...the floor? ...the back wall (or board)?



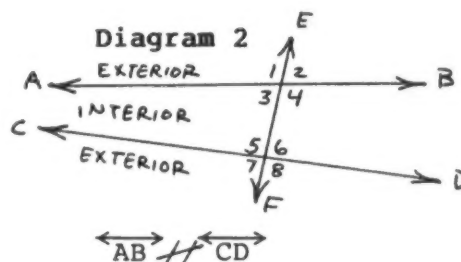
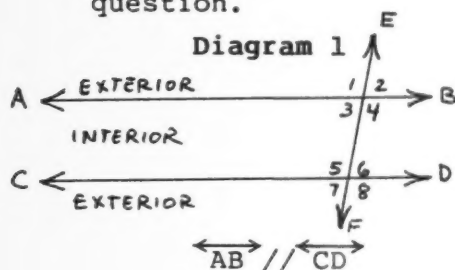
2. Ask: Which lines are parallel to "the crack" between the front board and floor (i.e., $\overleftrightarrow{AB}$)? (Elicit $\overleftrightarrow{DC}$, $\overleftrightarrow{GH}$ and $\overleftrightarrow{FE}$.) Since $\overleftrightarrow{FD}$ and $\overleftrightarrow{AB}$ do not meet, are they parallel? (No, since they do not lie in the same plane.)

3. The above motivates the following definitions:

- PARALLEL LINES are lines that lie in the same plane and have NO points in common (or ALL points in common). We write $\overleftrightarrow{AB} // \overleftrightarrow{CD}$.
- PARALLEL LINE SEGMENTS are segments in the same plane that do not intersect NO MATTER how far they are extended.
- SKEW LINES are two lines that do not lie in the same plane.

DEVELOPMENT:

- We would like to know what condition(s) MUST hold for two lines to be parallel (other than those spelled out in the definition). Let's draw two diagrams to study the above question.



(Using different colors, draw $\overleftrightarrow{EF}$ intersecting $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ in both diagrams.)

- In order to study this, we use an extra line $\overleftrightarrow{EF}$ forming eight angles in each case.

- $\overleftrightarrow{EF}$ is called a TRANSVERSAL. It cuts two lines in two points.
- $\angle 1$, $\angle 2$, $\angle 7$ and $\angle 8$ are called EXTERIOR angles.
- $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ are called INTERIOR angles.
- $\angle 3$ and $\angle 6$ are called ALTERNATE INTERIOR angles.

(Also $\angle 4$ and $\angle 5$. Note that they are interior angles on opposite sides of the transversal recognizable by forming a "Z" or "N" in various positions:

Z $\searrow$ N $\swarrow$)

- $\angle 3$ and $\angle 5$ are called INTERIOR ANGLES ON THE SAME SIDE OF THE TRANSVERSAL (or "consecutive interior angles"). (Also $\angle 4$ and $\angle 6$.)

f) $\angle 1$ and $\angle 5$ are called CORRESPONDING ANGLES. (They are on the same side of the transversal, one interior near one line and one exterior near the other line.) We recognize them by looking for an "F" in various positions: F, E , J , 7)

3. Today we will focus on ALTERNATE INTERIOR angles. Using the two diagrams, and your own, what relationship must there be between a pair of alternate interior angles in order for two lines to turn out to be parallel?

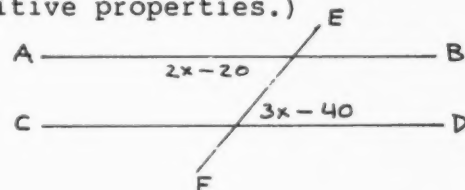
Elicit: If two lines are cut by a transversal so that the alternate interior angles are

- a) congruent, then the lines are parallel.
- b) not congruent, then the lines are NOT parallel.

We will use a) above as a basic postulate for proving lines are parallel.

APPLICATIONS:

1. Is parallelism an equivalence relation? (Yes, recall reflexive, symmetric and transitive properties.)
2. Prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$,
if $x=20$



3. How many lines can pass through P and be parallel to $\overleftrightarrow{AB}$? (one)



This question, simple as it may seem, is at the heart of an age old controversy. Many mathematicians thought it could be proved but NO one could do it. In fact, in the last century, mathematicians arrived at interesting results by assuming two or more lines can be drawn parallel to $\overleftrightarrow{AB}$ through P. However, here we will assume:

Euclid's Parallel Postulate: Through a given point NOT on a given line, there exists EXACTLY ONE line parallel to the given line.

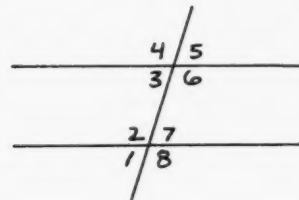
SUMMARY:

How can we prove that two lines are parallel?

HOMEWORK:

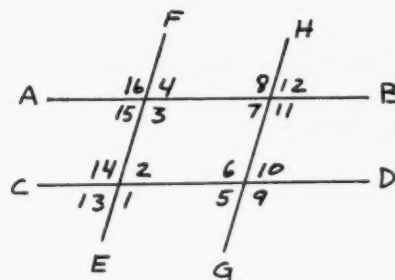
1. Name the type of angles described:

- $\angle 1$ and $\angle 3$
- $\angle 2$ and $\angle 6$
- $\angle 7$ and $\angle 6$
- $\angle 2$ and $\angle 3$
- $\angle 5$ and $\angle 7$

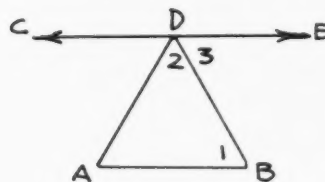


2. State which lines MUST be parallel. Given:

- $\angle 6 \cong \angle 11$
- $\angle 2 \cong \angle 5$
- $\angle 16 \cong \angle 11$
- $\angle 1 \cong \angle 6$



3. Given: $\overline{AD} \cong \overline{AB}$
 $\overline{DB}$ bisects $\angle ADE$
 Prove: $\overline{AB} \parallel \overline{CE}$



HOMEWORK SOLUTIONS:

- corresponding angles
 - alternate interior angles
 - interior angles on the same side of the transversal
 - interior angles on the same side of the transversal
 - corresponding angles
- $\overline{AB} \parallel \overline{CD}$
 - $\overline{EF} \parallel \overline{GH}$
 - $\overline{EF} \parallel \overline{GH}$
 - $\overline{EF} \parallel \overline{GH}$
- Since $\angle 1 \cong \angle 2$ (base angles of an isosceles triangle are congruent) and $\angle 2 \cong \angle 3$ (definition of bisector), we have $\angle 1 \cong \angle 3$ (transitive property of equality). Thus, a pair of alternate interior angles are congruent.

Lesson 36 - Parallelism

AIM: How do we prove that lines are parallel?

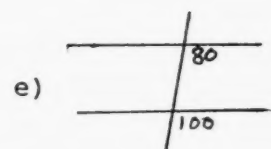
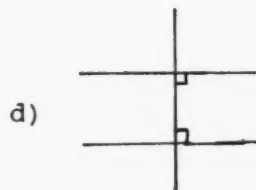
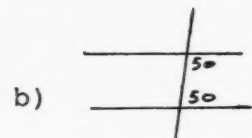
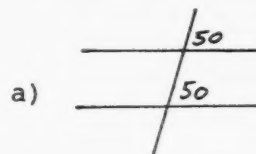
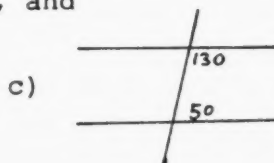
PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove informally that two lines cut by a transversal are parallel if:
 - a) a pair of corresponding angles are congruent.
 - b) a pair of interior angles on the same side of the transversal are supplementary.
 - c) they are perpendicular to the same line.
 - d) they are parallel to the same line.
2. Apply the five methods of proving lines parallel to numerical and algebraic problems.
3. Construct a line parallel to a given line through a given external point.
4. Apply the five methods of proving lines parallel to original proofs.

VOCABULARY: alternate interior angles, corresponding angles

MOTIVATION:

1. In each case determine whether or not the 2 lines are parallel, using the given information, and explain.



(In each case show whether or not a pair of alternate interior angles are congruent.)

Answers: a) yes b) no c) yes d)yes e) no

2. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF} \parallel \overleftrightarrow{CD}$, what relationship is there between $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$?

DEVELOPMENT:

The above problems informally lead to the following (elicit):

1. THEOREMS 1: Two lines cut by a transversal are parallel if
 - a) A pair of corresponding angles are congruent.
 - b) A pair of interior angles on the same side of the transversal are supplementary.
2. THEOREMS 2: Two lines are parallel if they are
 - a) Perpendicular to the same line (transversal) (see d).
 - b) Parallel to the same line (see 2).

APPLICATIONS:

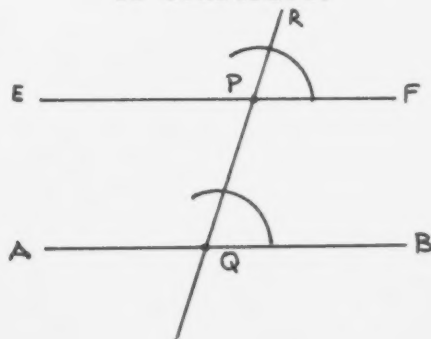
1. How can we construct a line through a given point PARALLEL to a given line?

Elicit the need for

- a) any transversal $\overleftrightarrow{PQ}$.
- b) constructing an angle at P so that we have congruent corresponding or alternate interior angles.

Discuss: Congruent corresponding angles are easier to construct.

2.



Result: $\overleftrightarrow{EF} \parallel \overleftrightarrow{AB}$

P

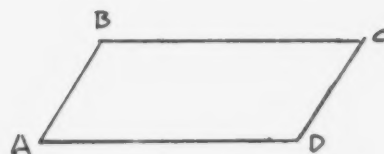
A ————— B

Write on Board

Procedure for constructing a line through a given point parallel to a given line.

1. Through P draw any transversal $\overleftrightarrow{PQ}$.
2. At P, construct $\angle RPF$ congruent to $\angle PQB$.
3. Draw $\overleftrightarrow{EF}$.

3. If $m\angle A = 60$ and $m\angle D = 120$, which lines (if any) are parallel?

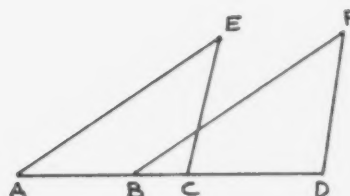


4. Given: $\overline{AB} \cong \overline{CD}$,

$$\overline{CE} \cong \overline{DF},$$

$$\angle ACE \cong \angle BDF$$

Prove: $\overleftrightarrow{AE} \parallel \overleftrightarrow{BF}$



5. Given: $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$

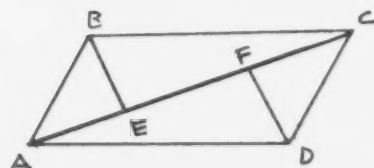
$$\overleftrightarrow{DF} \perp \overleftrightarrow{AC}$$

$$\overline{AF} \cong \overline{CE}, \overline{BE} \cong \overline{DF}$$

Prove: a) $\overleftrightarrow{BE} \parallel \overleftrightarrow{DF}$

b) $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$

c) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



SUMMARY:

How can we now prove that two lines are parallel? (Review all five methods.)

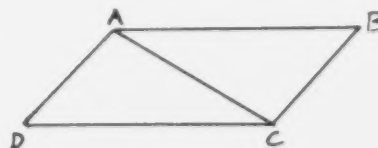
HOMEWORK:

1. Through P construct a line parallel to $\overleftrightarrow{AB}$.



2. Given $\overleftrightarrow{BC} \perp \overleftrightarrow{AC}$ and $\overleftrightarrow{DA} \perp \overleftrightarrow{AC}$, which lines MUST be parallel?

ex. 2-4



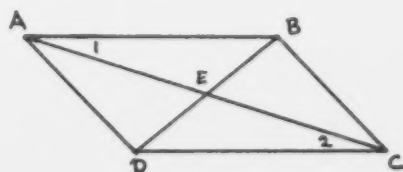
3. Given $m\angle B = 30$, $m\angle ACB = 70$, $m\angle DCA = 80$, which lines MUST be parallel?
4. Given $m\angle DAB = 120$, $m\angle BAC = 40$, $m\angle ACB = 80$, which lines MUST be parallel?

5. Given: $\angle 1 \cong \angle 2$

E is the midpoint of $\overline{AC}$.

Prove: a) $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$

b) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$



HOMEWORK SOLUTIONS:

1. See construction in lesson.
2. $\overleftrightarrow{AD} // \overleftrightarrow{BC}$
3. $\overleftrightarrow{AB} // \overleftrightarrow{DC}$
4. $\overleftrightarrow{AD} // \overleftrightarrow{BC}$
5. a) Since $\angle 1 \cong \angle 2$, we have $\overleftrightarrow{AB} // \overleftrightarrow{DC}$.
b) First prove $\triangle ABE \cong \triangle CDE$ by ASA $\cong$ ASA.
Then prove $\triangle DEA \cong \triangle BEC$ by SAS $\cong$ SAS.
Therefore $\angle DAE \cong \angle BCE$ and $\overleftrightarrow{AD} // \overleftrightarrow{BC}$.

Lesson 37 - Parallelism

AIM: What are the properties of parallel lines?

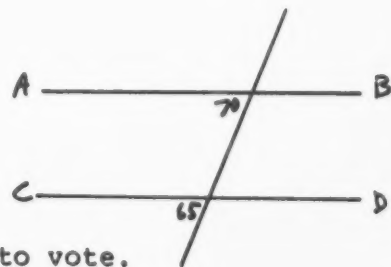
PERFORMANCE OBJECTIVES: The student will be able to...

1. State the properties of parallel lines. If two lines are parallel, then
 - a) corresponding angles are congruent.
 - b) alternate interior angles are congruent.
 - c) the interior angles on the same side of the transversal are supplementary.
 - d) a line perpendicular to one of them is perpendicular to the other.
2. Arrange these statements logically (postulate one, prove others informally).
3. Apply properties of parallel lines in numerical problems, algebraic problems, and proofs.

VOCABULARY: contrapositive

MOTIVATION:

1. Determine whether or not $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and explain.
2. Write the contrapositive of:
If you are 18, then you are old enough to vote.



DEVELOPMENT:

1. In ex. 2 recall: A statement and its contrapositive are equivalent (have the same truth values).
2. In ex. 1: We use the postulate from lesson 35: If two lines are cut by a transversal so that the alternate interior angles are NOT congruent, then the lines are not parallel.
What is the contrapositive of the above statement?

Thus, we have: If two lines cut by a transversal are parallel, then the alternate interior angles are congruent.

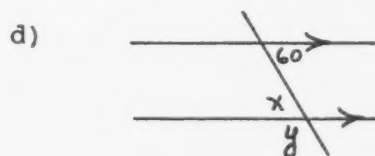
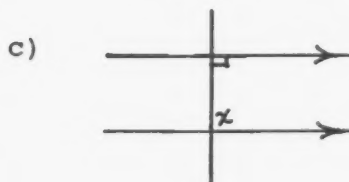
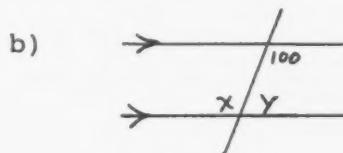
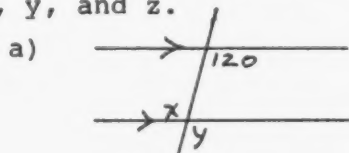
3. What is the difference between this result and previous work?

Previously: As a result of congruent alternate interior angles, the lines were parallel.

Now: As a result of lines being parallel, the alternate interior angles are congruent.

APPLICATIONS:

1. In each case, we are given a pair of parallel lines (as marked with arrows). Find x or find x and y or find x , y , and z .



2. The above informally proves the following. Elicit:

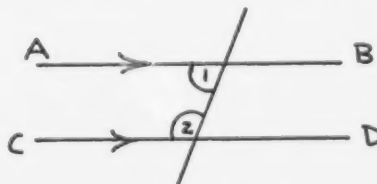
Theorem 1: If two parallel lines are cut by a transversal, then

- a) the corresponding angles are congruent (see 1a).
- b) a pair of interior angles on the same side of the transversal are supplementary (see 1b).

Theorem 2: If a line is perpendicular to one of two parallel lines, it is perpendicular to the other (see 1c).

3. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
 $m\angle 1 = 3x - 10$
 $m\angle 2 = 2x + 40$

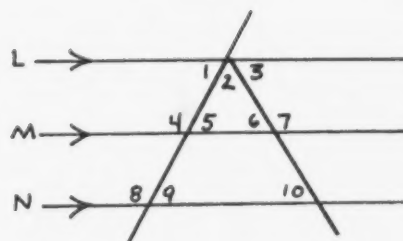
Find: a) x
 b) $m\angle 1$
 (30, 80)



4. Given: $\overleftrightarrow{m} \parallel \overleftrightarrow{n}$

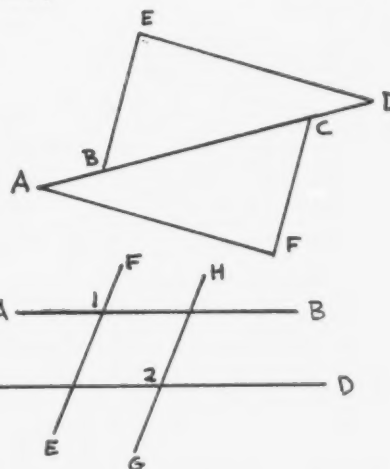
$$\angle 1 \cong \angle 2 \cong \angle 3$$

Find: $m\angle 1, m\angle 2, m\angle 3, m\angle 4,$
 $m\angle 5, m\angle 6, m\angle 7, m\angle 8,$
 $m\angle 9, m\angle 10$
 (60, 60, 60, 120,
 60, 60, 120, 120,
 60, 60)



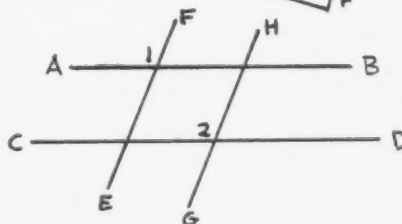
5. Given: $\overline{AB} = \overline{DC}$
 $\overleftrightarrow{AF} \parallel \overleftrightarrow{DE},$
 $\overleftrightarrow{BE} \parallel \overleftrightarrow{CF}$

Prove: $\overline{BE} \cong \overline{CF}$



6. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
 $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

Prove: $\angle 1 \cong \angle 2$



SUMMARY:

What information do we now have when two lines are known to be parallel?

HOMEWORK:

1. Given: $\angle 1 \cong \angle 2$

Prove: $\angle 3 \cong \angle 4$

2. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$$m\angle 4 = 2x$$

$$m\angle 5 = 3x - 10$$

Find: a) x b) $m\angle 5$

3. Given: $\angle A \cong \angle F$

$$\overleftrightarrow{AB} \parallel \overleftrightarrow{FE}$$

$$\overline{AB} \cong \overline{FE}$$

Prove: $\overleftrightarrow{AC} \parallel \overleftrightarrow{FD}$

4. Given: $\overleftrightarrow{AC} \parallel \overleftrightarrow{FD}$

$$\overline{BD} \cong \overline{EC}, \overleftrightarrow{AB} \parallel \overleftrightarrow{FE}$$

Prove: $\angle A = \angle F$

ex. 1-2



ex. 3-4



HOMEWORK SOLUTIONS:

1. Since $\angle 1 \cong \angle 2$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\angle 3 \cong \angle 4$

2. a) $x = 38$ b) 104

3. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{FE}$, we have $\angle B \cong \angle E$.
Also $\angle A \cong \angle F$ and $\overline{AB} \cong \overline{FE}$ (given).
 $\triangle ABC \cong \triangle FED$ by ASA $\cong$ ASA.
Thus $\angle ACB = \angle FDE$ and
 $\angle ACD \cong \angle FDC$ so $\overleftrightarrow{AC} \parallel \overleftrightarrow{FD}$.

4. Since $\overleftrightarrow{AC} \parallel \overleftrightarrow{FD}$, $\angle ACD \cong \angle FDC$, so $\angle ACB \cong \angle FDE$.
 $\overleftrightarrow{AB} \parallel \overleftrightarrow{FE}$, $\angle B \cong \angle E$
And since $\overline{BD} = \overline{EC}$, we have $\overline{BC} = \overline{DE}$.
Thus by ASA $\cong$ ASA, the two triangles are congruent and we
have $\angle A \cong \angle F$.

Lesson 38 - Parallelism

AIM: What is the sum of the measures of the angles of a triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove that the sum of the measures of the angles of a triangle is 180.
2. State five corollaries of this theorem.
3. Apply the theorem and its corollaries in numerical and algebraic problems.
4. Apply the theorem and its corollaries in original "proofs".

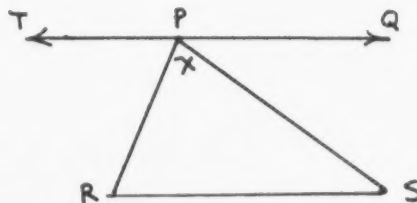
VOCABULARY: corollary

MOTIVATION:

Given: $\overleftrightarrow{TPQ} \parallel \overleftrightarrow{RS}$

$$m\angle TPR = 70, m\angle QPS = 45$$

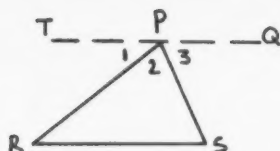
1. Find: a) $m\angle x$ b) $m\angle R$ c) $m\angle S$
2. What result concerning triangles does example 1 demonstrate?



DEVELOPMENT:

1. The above problem demonstrates that the SUM of the measures of the angles of a triangle is 180.
Ask: How do you know this result? Probably a teacher in an earlier grade told you so. Today we will actually prove this result. (The teacher can use EITHER of the following proofs, depending on the nature of the class. In either case, the basis of the proof is the first problem from today's lesson.)

2.



PROOF #1

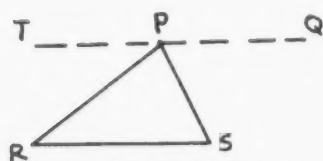
Given: $\triangle PRS$

Prove:

$$m\angle R + m\angle RPS + m\angle S = 180$$

- 1) Through P draw $\overleftrightarrow{TPQ} // \overleftrightarrow{RS}$.
- 2) $m\angle 1 = m\angle R$ and $m\angle 3 = m\angle S$ (Why?)
- 3) $m\angle 1 + m\angle 2 + m\angle 3 = 180$ (Why?)
- 4) $m\angle R + m\angle 2 + m\angle S = 180$.

PROOF #2



Given: $\triangle PRS$

Prove:

$$m\angle R + m\angle RPS + m\angle S = 180$$

Statements

Reasons

- | | |
|---|--|
| <p>1) Through P draw
$\overleftrightarrow{TPQ} // \overleftrightarrow{RS}$</p> | <p>1) Through a given point, not on a given line there is exactly one line parallel to the given line.</p> |
| <p>2) $m\angle TPR = m\angle R$,
$m\angle QPS = m\angle S$</p> | <p>2) If 2 lines are $//$, then alternate interior angles are congruent.</p> |
| <p>3) $m\angle TPR + m\angle RPS + m\angle QPS = 180$</p> | <p>3) The sum of the measures of all the angles about a given point on one side of a given straight line is 180.</p> |
| <p>4) $m\angle R + m\angle RPS + m\angle S = 180$</p> | <p>4) Substitution.
$m\angle R$ for $m\angle TPR$ and $m\angle S$ for $m\angle QPS$.</p> |

APPLICATIONS:

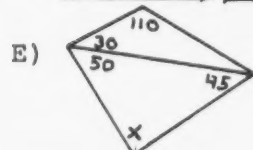
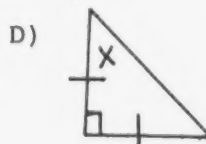
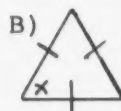
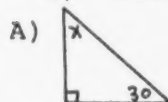
1. This theorem is one of the most important facts of mathematics. As a result of this theorem, several other theorems stem directly.

Definition: A COROLLARY is a theorem that follows easily from another theorem.

2. For each diagram,

a) find $m\angle X$

b) state a result using the theorem.



Ex. A) leads to

COROLLARY 1: The two acute angles of a right triangle are complementary.

Ex. B) leads to

COROLLARY 2: Each angle of an equilateral triangle measures 60.

Ex. C) leads to

COROLLARY 3: If two angles of one triangle are congruent to two angles of a second triangle, then the third angles of the triangles are also congruent.

Ex. D) leads to

COROLLARY 4: Each acute angle of an isosceles right triangle measures 45.

Ex. E) leads to

COROLLARY 5: The sum of the measures of the angles of a quadrilateral is 360.

3. For each statement give an EXPLANATION (informal proof) why the statement IS or IS NOT true.

- A triangle can have two right angles.
- A right triangle can have two congruent angles.
- A triangle can have two obtuse angles.
- An isosceles triangle can have a right angle.

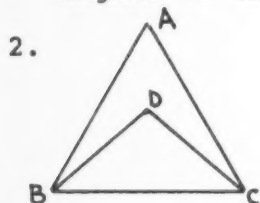
4. The measure of the vertex angle of an isosceles triangle is 30 less than the measure of a base angle. Find the measure of all three angles.

SUMMARY:

State a geometric result which depends on the "angle-sum" theorem learned today.

HOMEWORK:

1. Each base angle of an isosceles triangle measures 15 more than the vertex angle. Find the measures of the three angles of the triangle.

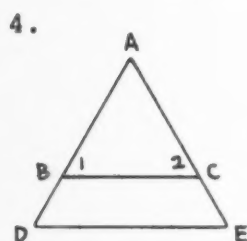


Given: $m\angle A = 80$, $\overline{AB} \cong \overline{AC}$

$\overrightarrow{BD}$ and $\overrightarrow{CD}$ bisect $\angle ABC$ and $\angle ACB$ respectively.

Find: $m\angle D$

3. Find the number of degrees in each angle of a triangle whose angles have measures in the ratio 2:3:4.



Given: $\overline{AB} \cong \overline{AC}$, $\overline{BD} \cong \overline{CE}$

Prove: $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$

HOMEWORK SOLUTIONS:

1. $x + (x + 15) + (x + 15) = 180$ so the three angles measure 50, 65 and 65.

2. 130

3. 40, 60, 80

4. The given leads to two isosceles triangles, $\triangle ABC$ and $\triangle ADE$ with $\angle 1 \cong \angle 2$ and $\angle D \cong \angle E$.

$$\text{Since } m\angle 1 + m\angle 2 + m\angle A = m\angle D + m\angle E + m\angle A = 180$$

$$m\angle 1 + m\angle 2 = m\angle D + m\angle E$$

$$2m\angle 1 = 2m\angle D$$

$$\text{So } \overleftrightarrow{BC} \parallel \overleftrightarrow{DE} \quad m\angle 1 = m\angle D$$

Lesson 39 - Parallelism

AIM: Can we prove triangles congruent if they agree in two angles and a side opposite one of these angles?

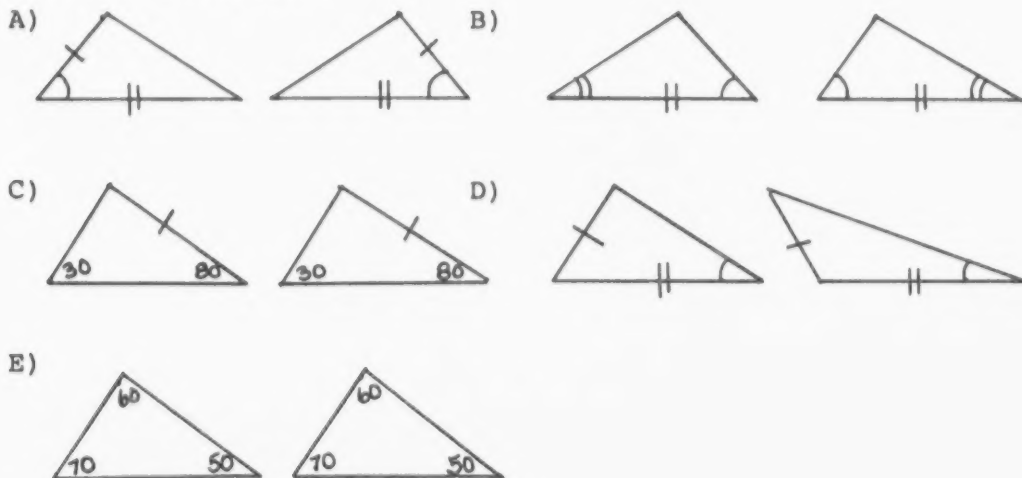
PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove informally that two triangles are congruent if they agree in two angles and a side opposite one of these angles.
2. State why $SSA \cong SSA$ and $AAA \cong AAA$ do not necessarily make two triangles congruent.
3. Apply the new method of congruent triangles in original proofs and numerical problems.

VOCABULARY: no new terms

MOTIVATION:

In each case, determine whether or not the two triangles are congruent and explain.



DEVELOPMENT:

1. Ex. A) and B) review the $SAS \cong SAS$ and the $ASA \cong ASA$ method of proving triangles congruent.

2. In ex. C) what given parts are congruent? ($SAA = SAA$)

This was not a congruence method previously learned.

Ask: Using recent work, ARE any other parts of these triangles congruent? (Elicit: Since two angles of one triangle are congruent to two angles of the other triangle, their third angles are also congruent.)

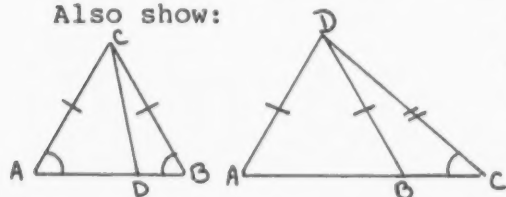
Note: We can now apply another method. Which one? ($ASA \cong ASA$) Thus we have the theorem:

Two triangles are congruent if they agree in two angles and a side opposite one of these angles.

($SAA \cong SAA$)

3. Ex. D) illustrates that if $SSA \cong SSA$, the two triangles DO NOT have to be congruent (you might wish to demonstrate this with a wooden model).

Also show:

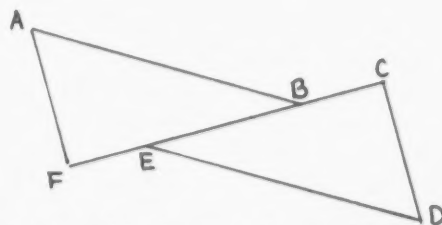


In these diagrams, $\triangle BDC$ and $\triangle ADC$ agree in SSA. However, they cannot be congruent.

4. Example E) illustrates that if $AAA = AAA$, the two triangles DO NOT have to be congruent.

APPLICATIONS:

1. Given: $\overline{AF}$ and $\overline{DC}$ are perpendicular to $\overleftrightarrow{FC}$.
 $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, $\overline{AF} \cong \overline{DC}$
 Prove: $\overline{BA} \cong \overline{ED}$



2. Prove that the altitudes to the legs of an isosceles triangle are congruent.

SUMMARY:

What are the FOUR BASIC METHODS of proving triangles congruent? ($SAS \cong SAS$, $ASA \cong ASA$, $SSS \cong SSS$, $SAA \cong SAA$)

HOMEWORK:

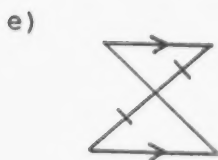
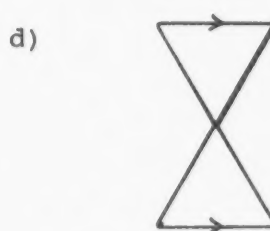
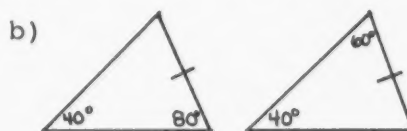
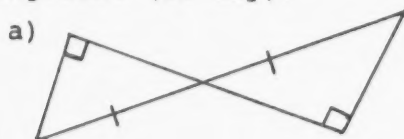
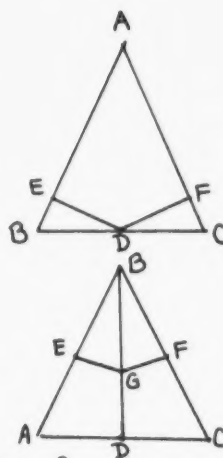
- Given: D is midpoint of $\overline{BC}$
 $\overline{AB} \cong \overline{AC}$
 $\overleftrightarrow{DE} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{DF} \perp \overleftrightarrow{AC}$

Prove: $\angle BDE = \angle CDF$

- Given: $\overleftrightarrow{BD}$ bisects $\angle ABC$
 $\overleftrightarrow{GE} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{GF} \perp \overleftrightarrow{CB}$

Prove: $\overline{EG} \cong \overline{FG}$

- State in words the result of ex. 2.
- State a method of proving that the two triangles are congruent (if any)?



HOMEWORK SOLUTIONS:

- Prove $\triangle BED \cong \triangle CFD$ by $SAA \cong SAA$.
- Prove $\triangle BEG \cong \triangle BFG$ by $SAA \cong SAA$.
- If a point lies on the bisector of an angle, it is equidistant from the sides of the angle.

4. a) $SAA \cong SAA$
 b) Either $SAA \cong SAA$ or $ASA \cong ASA$
 c) The two triangles are not necessarily congruent.
 d) The two triangles are not necessarily congruent.
 e) Either $ASA \cong ASA$ or $SAA \cong SAA$.

Lesson 40 - Parallelism

AIM: Can we prove the converse of the isosceles triangle theorem?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State and informally prove using $SAA \cong SAA$, the converse of the isosceles triangle theorem.
2. Identify the congruent sides given two congruent angles in simple diagrams and in those with overlapping triangles.
3. State and prove the corollary, "If a triangle is equiangular, then it is equilateral."
4. Apply the theorem and its corollary in numerical and algebraic problems and in proofs.

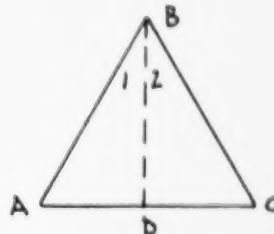
VOCABULARY: converse, equilateral triangle, equiangular triangle

MOTIVATION:

1. Write the converse of $p \rightarrow q$.
2. If $p \rightarrow q$ is true, is its converse also true? Explain.
3. Write the converse of: If two sides of a triangle are congruent, then the angles opposite these sides are also congruent.

DEVELOPMENT:

1. In 3 above, we have, "If two angles of a triangle are congruent, then the sides opposite these angles are also congruent."
2. Is this a "true" converse?
Let's explore by trying to prove it in a way similar to the way we proved the original theorem.



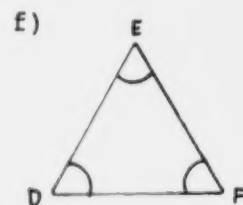
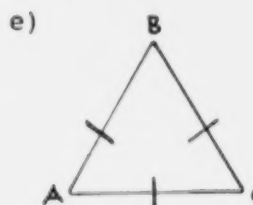
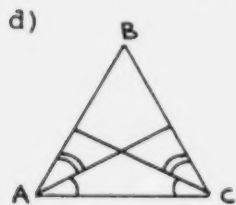
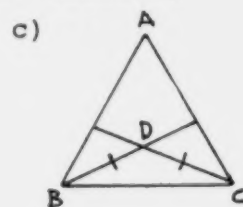
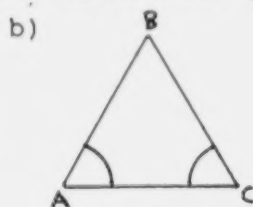
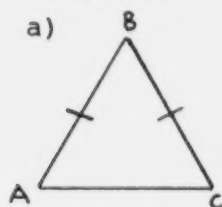
Given: $\angle A \cong \angle C$

Prove: $\overline{AB} \cong \overline{CB}$

- a) Draw $\overrightarrow{BD}$ bisecting $\angle ABC$.
- b) By $\text{SAA} \cong \text{SAA}$ the two triangles formed are congruent.
- c) $\overline{AB} \cong \overline{CB}$

Therefore the theorem and its converse are true. (An alternative way of stating the converse is, "If two angles of a triangle are congruent, then the triangle is isosceles.")

3. In each diagram NAME ALL other parts that MUST be congruent. State a reason for your results.

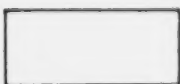


4. Ex. e) recalls: If a triangle is equilateral then it is also equiangular. (Recall the meanings of these words and the differences.)
5. Ex. f) leads to

COROLLARY: If a triangle is equiangular, then it is also equilateral.

6. Point out that "equiangular" and "equilateral" go together for triangles. (See 4 and 5 above.) Demonstrate that this is not true for quadrilaterals.

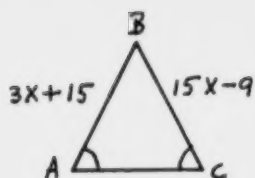
Show:



These figures demonstrate that you MUST be careful to distinguish between these NEW terms.

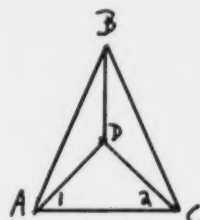
APPLICATIONS:

1. Find x , AB , CB .



2. How many degrees are contained in each angle of an equilateral triangle?

3. Given: $\angle 1 \cong \angle 2$
 $\angle ADB \cong \angle CDB$



Prove: $\angle BAC \cong \angle BCA$

SUMMARY:

1. What is the isosceles theorem?
2. What is its converse?
3. Explain these terms.
a) equilateral b) equiangular
4. Is an equilateral figure AUTOMATICALLY equiangular? Explain.
5. Is an equiangular figure AUTOMATICALLY equilateral? explain

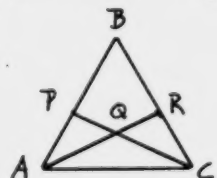
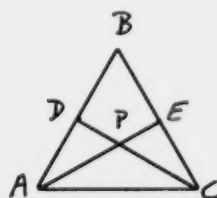
HOMEWORK:

1. Given: Equilateral $\triangle ABC$
 $\overrightarrow{AE}$ bisects $\angle BAC$,
 $\overrightarrow{CD}$ bisects $\angle BCA$

Find: a) $m\angle AEC$ b) $m\angle EPC$

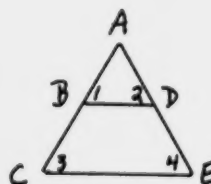
2. Given: $\overline{AB} \cong \overline{CB}$, $\overrightarrow{AR}$ bisects $\angle BAC$,
 $\overrightarrow{CP}$ bisects $\angle BCA$

Prove: a) $\overline{AP} \cong \overline{CR}$
b) $\triangle AQC$ is isosceles

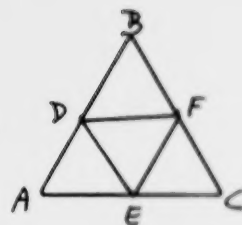


3. Given: $\overline{AC} \cong \overline{AE}$,
 $\overleftarrow{BD} \parallel \overleftarrow{CE}$

Prove: a) $\triangle ABD$ is isosceles
b) $\overline{BC} \cong \overline{DE}$



4. If $\triangle ABC$ is equilateral and $\triangle DEF$ is formed by connecting consecutive midpoints, find $m\angle EDF$, $m\angle EFD$, $m\angle DEF$.



HOMework SOLUTIONS:

1. a) 90 b) 60
2. a) First prove $\triangle ARC \cong \triangle CPA$ by ASA $\cong$ ASA. It follows that $\overline{AP} \cong \overline{CR}$.
b) Since $\angle QAC \cong \angle ACQ$, $\triangle AQC$ is isosceles.
3. a) Since $\overline{AC} \cong \overline{AE}$, $\angle 3 \cong \angle 4$
Since $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$, $\angle 1 \cong \angle 3$, $\angle 2 \cong \angle 4$
Thus by substitution, $\angle 1 = \angle 2$ and $\triangle ABD$ is isosceles.
b) $\overline{AC} \cong \overline{AE}$ (given) so $AC = AE$
 $\overline{AB} \cong \overline{AD}$ (see a) so $AB = AD$
Thus, $BC = DE$ so $\overline{BC} \cong \overline{DE}$
4. By the SAS method of congruence, $\triangle ADE \cong \triangle BDF \cong \triangle FCE$. All three are equilateral.
Thus, $\overline{DE} \cong \overline{EF} \cong \overline{FD}$ making $\triangle DEF$ equilateral, so its three angles each measure 60.

Lesson 41 - Congruence

AIM: How do we prove right triangles congruent?

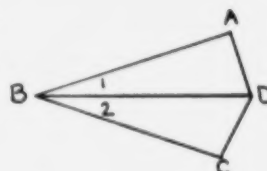
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define hypotenuse and leg of a right triangle.
2. State the theorem, "Two right triangles are congruent if they agree in the hypotenuse and a leg."
3. Apply the theorem in numerical and algebraic problems and in proofs.

VOCABULARY: hypotenuse, leg, right angle, right triangle

MOTIVATION

1. Given: $\overleftrightarrow{DA} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{DC} \perp \overleftrightarrow{CB}$
 $\overline{DA} \cong \overline{DC}$
Prove: $\angle 1 \cong \angle 2$



Pose: What methods of proving triangles congruent have we studied?

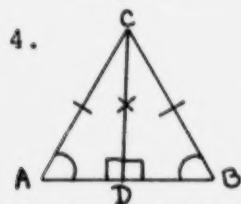
Elicit: $SAS \cong SAS$, $ASA \cong ASA$, $SSS \cong SSS$, $SAA \cong SAA$

Elicit also that the above problem does NOT fit into ANY of these categories. We have $SSA \cong SSA$!

2. Recall from lesson #39 this example which illustrates that although $SSA \cong SSA$, $\triangle ACD \not\cong BCD$.

3. $\overline{CD}$ was any line drawn to $\overline{AB}$.

Ask: How can we draw $\overline{CD}$ so that the two triangles are congruent? (Elicit: Draw $\overline{CD} \perp \overline{AB}$.)



4. This is the ONLY case in which $SSA \cong SSA$ makes the triangles congruent, when the triangles are RIGHT TRIANGLES. ($SAA \cong SAA$ results.)

5. Note that the congruent sides were the hypotenuses and ONE pair of legs.

DEVELOPMENT:

1. We state the previous result as a theorem.

Theorem: Two RIGHT triangles are congruent if the hypotenuse and leg of one triangle are congruent to the hypotenuse and leg of the other triangle.

2. With a good class only: Outline proof

Given: $\angle C$ and $\angle T$ are right $\angle$'s

$$\overline{AB} \cong \overline{RS}, \overline{AC} \cong \overline{RT}$$

Prove: $\triangle ABC \cong \triangle RST$

Plan: Construct $\triangle APB \cong \triangle RTS$,
then prove $\triangle ACB \cong \triangle APB$ by:

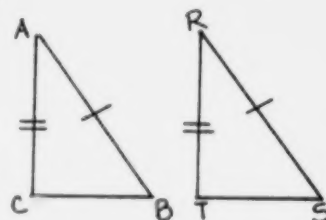
a) $\angle ACB \cong \angle APB$

b) $\angle 1 \cong \angle 2$

c) $\angle 3 \cong \angle 4$

d) $\overline{CB} \cong \overline{PB}$ and $\overline{AC} \cong \overline{AP}$

So SAS $\cong$ SAS.



APPLICATIONS:

1. Given: $\overleftrightarrow{PR} \perp \overleftrightarrow{BC}$, $\overleftrightarrow{QS} \perp \overleftrightarrow{BC}$
 $\overline{PB} \cong \overline{QC}$ and $\overline{BS} \cong \overline{CR}$

Prove: $\triangle ABC$ is isosceles.

2. Given: $\overleftrightarrow{EF} \perp \overleftrightarrow{AD}$, $\overleftrightarrow{BC} \perp \overleftrightarrow{AD}$
 $\overline{AB} \cong \overline{DE}$, $\overline{AF} \cong \overline{DC}$

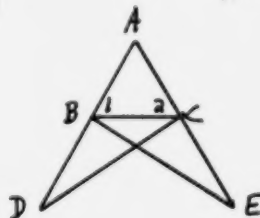
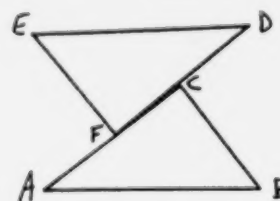
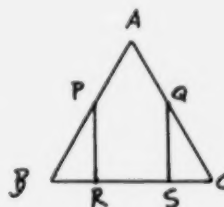
Prove: a) $\overline{EF} \cong \overline{BC}$

b) $\overleftrightarrow{ED} \parallel \overleftrightarrow{BA}$

3. Given: $\angle A$ is a right $\angle$
 $\overline{DC} \cong \overline{EB}$,

$\angle 1 \cong \angle 2$

Prove: $\angle D \cong \angle E$



SUMMARY:

Under what conditions will SSA = SSA guarantee that two triangles are congruent?

HOMEWORK:

1. Given: $\overleftrightarrow{CF} \perp \overleftrightarrow{BD}$, $\overleftrightarrow{AE} \perp \overleftrightarrow{BD}$

$$\overline{CD} \cong \overline{AB}, \overline{BF} \cong \overline{DE}$$

Prove: a) $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$

$$\text{b) } \overline{AD} \cong \overline{CB}$$

2. Given: $\overleftrightarrow{AD} \perp \overleftrightarrow{DE}$, $\overleftrightarrow{BE} \perp \overleftrightarrow{DE}$

$$\overline{AE} \cong \overline{BD}$$

Prove: a) $\triangle BDE \cong \triangle AED$

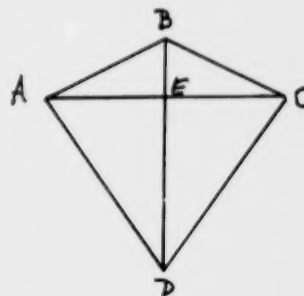
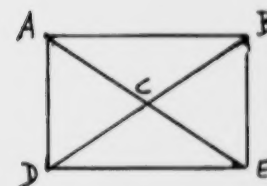
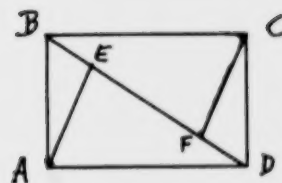
$$\text{b) } \triangle ADC \cong \triangle BEC$$

$$\text{c) } \triangle ACD \cong \triangle BCE$$

3. Given: $\overleftrightarrow{BA} \perp \overleftrightarrow{AD}$, $\overleftrightarrow{BC} \perp \overleftrightarrow{CD}$

$$\angle BAE \cong \angle BCE$$

Prove: $\angle ADE \cong \angle CDE$



HOMEWORK SOLUTIONS:

1. a) Prove $\triangle ABE = \triangle CDF$ by hyp. leg = hyp. leg.

Thus, $\angle ABE = \angle CDF$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

b) Now use $\overline{AE} \cong \overline{CF}$ to help get $\triangle ADE = \triangle CBF$ by SAS = SAS. Thus, $\overline{AD} = \overline{CB}$.

2. a) Prove $\triangle ADE = \triangle BED$ by hyp. leg = hyp. leg

so $\angle BDE \cong \angle AED$.

b) Comes from the subtraction postulate.

c) Use b) and SAA = SAA.

3. Since $\angle BAE \cong \angle BCE$, $\overline{BA} = \overline{BC}$.

Thus, using hyp. leg = hyp. leg,

we have $\triangle ABD = \triangle CBD$ and $\angle ADE = \angle CDE$.

Lesson 42 - Congruence

AIM: How is an exterior angle of a triangle related to the non-adjacent interior angles?

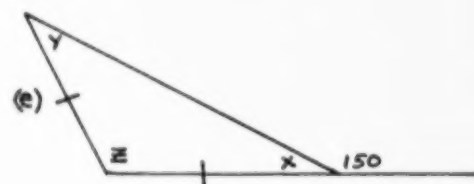
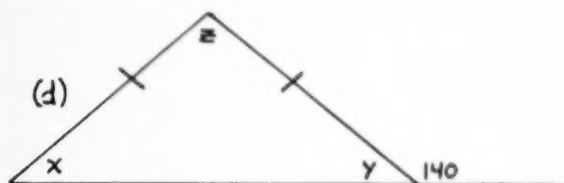
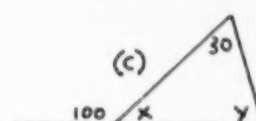
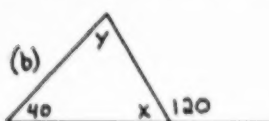
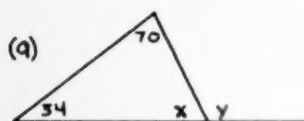
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define exterior angle of a triangle; non-adjacent angles.
2. State the relationship between an exterior angle and the adjacent interior angle.
3. State the theorem, "The measure of an exterior angle of a triangle is equal to the sum of the measures of the non-adjacent interior angles."
4. Apply the above theorem to numerical problems.

VOCABULARY: exterior angle of a triangle, remote interior angle, non-adjacent interior angle

MOTIVATION:

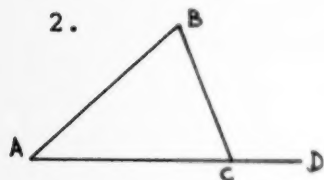
In each case, find the measures of the missing angles.



DEVELOPMENT:

1. What do the five diagrams have in common?

2.



Note that each diagram has an "exterior angle of a triangle" such as $\angle BCD$. Pose the question: How would you define exterior angle of a triangle?

Definition: An exterior angle of a triangle is an angle formed by one side of the triangle and a ray of another side.

- Note that $\angle BCA$ is the ADJACENT interior angle to exterior $\angle BCD$. Pose, "Why do we use this terminology?"
- What would be an appropriate name for $\angle A$ and $\angle B$? (Either "NON-ADJACENT Interior Angles" or "REMOTE Interior Angles" with respect to $\angle BCD$.)
- As a result of our five problems, can you state the relationship between an exterior angle of a triangle and
 - the adjacent interior angle (supplementary).
 - the remote interior angles.

Theorem: The measure of an exterior angle of a triangle is equal to the SUM of the measures of the two remote interior angles.

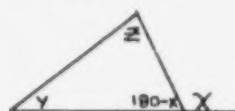
APPLICATIONS:

- Prove (algebraically) the above theorem.

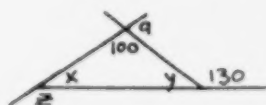
Show: $180 - x + y + z = 180$

$$y + z - x = 0$$

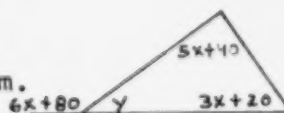
$$y + z = x$$



- Find x , y , z , a .



- Find x , y and the measures of the angles in the diagram.



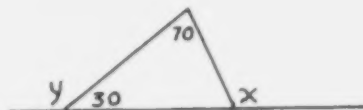
- If one remote interior angle of a triangle has a measure twice as large as the measure of the other remote interior angle, and the corresponding exterior angle of the triangle measures 150, find the measures of the three angles of the triangle.

SUMMARY:

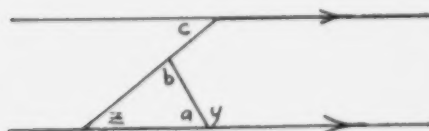
1. What is the relationship between an exterior angle of a triangle and
 - a) its adjacent interior angle of the triangle?
 - b) its remote interior angles of the triangle?

HOMEWORK:

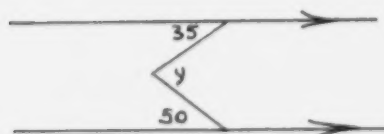
1. Find x , y .



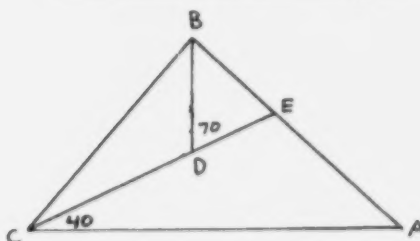
2. If $m\angle c = 70 + 2x$,
 $m\angle y = 120 - 3x$
 $m\angle b = 4x + 5$
 Find y .



3. Find y .



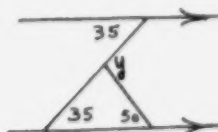
4. If $\overrightarrow{BD}$ bisects $\angle ABC$ and
 $\overrightarrow{CE}$ bisects $\angle BCA$,
 find $m\angle A$.



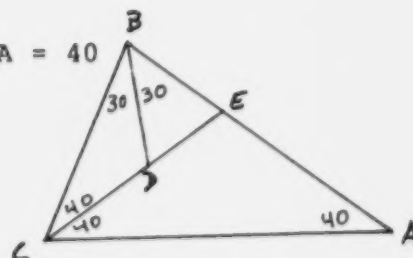
HOMEWORK SOLUTIONS:

1. $x = 100$, $y = 150$
2. $120 - 3x = 70 + 2x + 4x + 5$
 $x = 5$
 Thus, $m\angle z = 80$, $m\angle b = 25$, $y = 105$

3. $y = 85$



4. Ans. $m\angle A = 40$



Lesson 43 - Congruence

AIM: What are the formulas concerning the interior angles of a polygon?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define equilateral polygon, equiangular polygon and regular polygon.
2. Name polygons which have 4, 5, 6, 7, 8, 9 and 10 sides.
3. State formulas for
 - a) the sum of the measures of the interior angles of a polygon.
 - b) one interior angle of a regular n -gon.
4. Apply the formulas in numerical and algebraic problems.

VOCABULARY: equilateral, equiangular, regular polygon, disjoint triangles

MOTIVATION:

1. a) If the sides of a polygon are congruent, are the angles also? Explain.
- b) If the angles of a polygon are congruent, are the sides also? Explain.

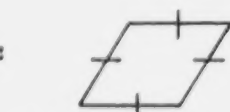
The above lead to the following three definitions.

DEFINITIONS:

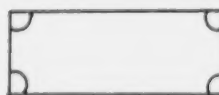
1. An EQUILATERAL polygon is one with all of its SIDES congruent.
2. An EQUIANGULAR polygon is one with all of its ANGLES congruent.
3. If a polygon is BOTH equilateral AND equiangular then it is said to be a REGULAR POLYGON.

EXAMPLES:

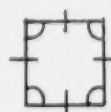
(4-sided figures):



equilateral



equiangular



regular

2. What is the SUM of the measures of the angles of a quadrilateral? (Elicit 360 - Ask why.) Demonstrate:

A quadrilateral can be divided into two DISJOINT (non-overlapping)



triangles, each with an angle sum of 180. Thus, we get a total of 360.

DEVELOPMENT:

Develop the following chart (using diagrams where needed).

# OF SIDES IN POLYGON	NAME OF POLYGON	# OF DISJOINT △'S FORMED BY ALL THE DIAGONALS FROM ONE VERTEX	SUM OF M'S OF △'S IN POLYGON	M OF EACH △ IF THE POLYGON IS REGULAR
3	triangle	1	180	60
4	quadrilateral	2	$2(180)=360$	90
5	pentagon	3	$3(180)=540$	108
6	hexagon	4	$4(180)=720$	120
7	heptagon	5	$5(180)=900$	$128\frac{4}{7}$
8	octagon	6	$6(180)=1080$	135
9	nonagon	7	$7(180)=1260$	140
10	decagon	8	$8(180)=1440$	144
11	undecagon	9	$9(180)=1620$	$147\frac{3}{11}$
12	dodecagon	10	$10(180)=1800$	150
20	icosagon	18	$18(180)=3240$	162
⋮	⋮	⋮	⋮	⋮
n	n-gon	$n - 2$	$180(n - 2)$	$\frac{180(n-2)}{n}$

The above leads to the following:

THEOREMS: In an n-gon,

- $n - 2$ disjoint triangles can be formed using all the diagonals from one vertex
- the SUM of the measures of the INTERIOR angles is $180(n - 2)$
- if it is a regular n-gon, EACH interior angle measures $\frac{180(n-2)}{n}$

APPLICATIONS:

1. Find the sum of the measures of the interior angles of a 52 sided polygon. (9000)
2. The sum of the measures of the interior angles of a polygon is 3600. Find the number of sides in this polygon. (22)
3. If the measure of each interior angle of a regular polygon is 170, find the number of sides. (36)

SUMMARY:

1. Under which circumstances do we use the formula
 $180(n - 2)$?
 $\frac{180(n-2)}{n}$?
2. Why don't we divide out the n's in the second formula?

HOMEWORK:

1. Find the sum of the measures of the angles of a 75 sided polygon.
2. How many disjoint triangles can be formed drawing all the diagonals from one vertex of a 1000 sided convex polygon?
3. What is the measure of each interior angle of a regular 200 sided polygon?
4. Which equilateral polygons are AUTOMATICALLY equiangular.
5. The sum of the measures of the interior angles of a certain polygon is 720. Find the sum of the measures of the interior angles of a polygon with two more sides than the first.
6. Each interior angle of a regular polygon measures 178. Find the number of sides in the polygon.
7. Redo example 6 if each angle measures 80.

HOMEWORK SOLUTIONS:

1. $180(73) = 13,140$

2. 998

3. $\frac{180(198)}{200} = \frac{35640}{200} = 178.2$

4. equilateral triangles

5. $180(n - 2) = 720$		720
$n - 2 = 4$	OR	<u>+360</u>
$n = 6$		1080
$180(6) = 1080$		

6. $\frac{180(n-2)}{n} = 178$

$$180n - 360 = 178n$$

$$2n = 360$$

$$n = 180$$

7. No such polygon because

$$\frac{180(n-2)}{n} = 80$$

$$180n - 360 = 80n$$

$$n = 3.6$$

n MUST be an integer!

Lesson 44 - Congruence

AIM: What are the formulas concerning the exterior angles of a triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State how changing the number of sides of a polygon affects the
 - a) sum of the measures of the interior angles
 - b) the measure of each interior angle if the polygon is regular
 - c) the measure of each exterior angle if the polygon is regular
2. State the theorem concerning the sum of the exterior angles of a convex polygon.
3. State the formula for one exterior angle of a regular n -GON.
4. Apply the theorem and formula to numerical and algebraic problems

VOCABULARY: exterior angle of a polygon

MOTIVATION:

1. Find the measure of each EXTERIOR ANGLE of a regular octagon. (Recall measuring of exterior angle, previously studied for triangles.)
2. Fill in the information in this chart from yesterday's work adding an extra column. (Refer to yesterday's chart as well.)

NUMBER OF SIDES	SUM OF M'S OF INTERIOR $\angle$ 'S	M OF EACH INTERIOR $\angle$ IF FIG. IS REG.	M OF EACH EXTERIOR $\angle$ IF FIG. IS REG.
3	180	60	120
6	720	120	60
8	1080	135	45
10	1440	144	36
20	3240	162	18

3. Let us make observations from this chart.

As the number of sides of an equiangular polygon INCREASES:

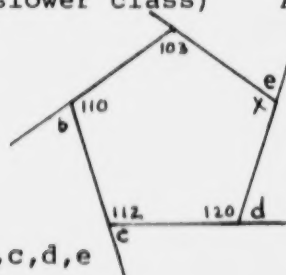
- a) The sum of the measures of the INTERIOR angles INCREASES.
 - b) The measure of each INTERIOR angle INCREASES.
 - c) The measure of each EXTERIOR angle DECREASES.
- (Explain why, intuitively!)

What is true about the SUM of the m's of the exterior angles?

DEVELOPMENT:

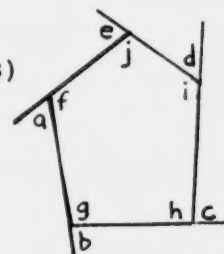
1. SURPRISE! It stays constant. (Show for each of the cases above.)
2. What is true if the figure is NOT regular (but convex)? Will the sum still be 360? Use one of the following approaches for a PENTAGON.

APPROACH A (slower class)



- a) Find x
- b) Find a, b, c, d, e
- c) Find $a+b+c+d+e$

APPROACH B (better class)



- a) $a+f+b+g+c+h+d+i+e+j = 180(5) = 900$
 - b) $f+g+h+i+j = 540$
- subtraction postulate gives
 $a+b+c+d+e = 360$

This DEMONSTRATES that for a pentagon, the sum of the m's of the exterior angles, one $\angle$ per vertex, is 360.

This PROVES that for a

3. We can show that this holds true for other polygons similarly. Thus we have:

THEOREM: The sum of the measures of the exterior angles of a convex polygon, one angle at each vertex, is 360.

APPLICATIONS:

1. Find the number of degrees in the measure of each interior angle of a regular pentagon.

METHOD 1 $\frac{180(5-2)}{5} = 108$

METHOD 2: Each exterior angle measures $\frac{360}{5} = 72$. Thus, each interior angle measures $180 - 72 = 108$.

Note: With this new method, you must remember to take the SUPPLEMENT at the end.

2. Find the number of sides in a regular polygon whose exterior angle measures 40. (9)
3. Each interior angle of a regular polygon has a measure of 170. How many sides are there in the n-gon? (36)

SUMMARY:

1. What is the SUM of the interior angles of an n-gon?
2. What is the SUM of the exterior angles of a convex n-gon?
3. What is the measure of each interior angle of a regular n-gon?
4. What is the measure of each exterior angle of a regular n-gon?

HOMEWORK:

1. Find the measure of each EXTERIOR angle of an equiangular icosagon.
2. Use your results in 1) to find the measure of each interior angle of an equiangular icosagon.
3. Redo example 2) using another method.
4. Find the number of sides in a regular polygon, each of whose exterior angles measures 5.
5. The sum of the interior angles of a regular n-gon is 3060. Find the number of degrees in the measure of an exterior angle of a regular polygon with one more side than this n-gon.
6. What is the largest measure an exterior angle of an equiangular polygon can have?

7. What is the smallest measure an exterior angle of an equiangular polygon can have?

HOMEWORK SOLUTIONS:

1. $\frac{360}{20} = 18$

2. 162

3. $\frac{(20-2)180}{20} = 18(9) = 162$

4. $\frac{360}{5} = 72$

5. $3060 = 180(n-2)$

$$\frac{360}{20} = 18$$

$$180n = 3420$$

$$n = 19$$

Since the measure of the exterior angle decreases as the number of sides increases

6. the largest measure occurs for a triangle. (120)

7. there is no smallest! As $n \rightarrow \infty$, $\frac{360}{n} \rightarrow 0$.

With a very large number of sides, the angle is close to 0.

Lesson 45 - Quadrilaterals

AIM: What are the properties of a parallelogram?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define parallelogram.
2. State the theorem, "A diagonal divides a parallelogram into two congruent triangles."
3. State four corollaries to this theorem:
 - a) opposite sides congruent
 - b) opposite angles congruent
 - c) consecutive angles supplementary
 - d) diagonals bisect each other
4. Apply the properties of a parallelogram in numerical and algebraic problems.
5. Apply the properties of a parallelogram in writing original proofs.

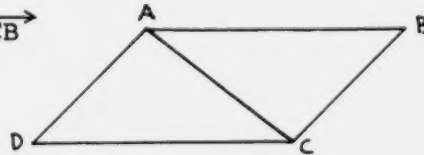
VOCABULARY: consecutive angles, parallelogram

MOTIVATION:

1. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$

Prove:

- a) $\triangle ADC \cong \triangle CBA$
- b) What parts of these triangles are congruent?



2. In the above problem we have a special 4-sided figure (quadrilateral).

Definition: A PARALLELOGRAM is a quadrilateral in which the opposite sides are parallel. We use the symbol $\square$ ABCD or $\square$ ABCD.

Note that we name a $\square$ using the consecutive letters going about the figure counter clockwise or clockwise.

3. From the previous problem, as well as properties of parallel lines, we have

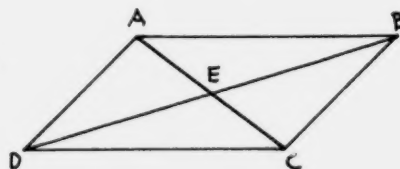
THEOREMS: PROPERTIES OF A PARALLELOGRAM

1. A diagonal of a $\square$ divides it into two congruent triangles.
2. The opposite sides of a $\square$ are congruent.
3. The opposite angles of a $\square$ are congruent.
4. Consecutive (successive) angles of a $\square$ are supplementary.

DEVELOPMENT:

Given: $\square ABCD$

- a) Which triangles do you believe to be congruent? Prove it!
(Take either $\triangle ABE \cong \triangle CDE$ or $\triangle DEA \cong \triangle BEC$)



- b) Using corresponding sides of congruent triangles are congruent, we have just proved another property of parallelograms.

Theorem: 5. The diagonals of a $\square$ bisect each other.

APPLICATIONS:

Let's use the new theorems in these problems.

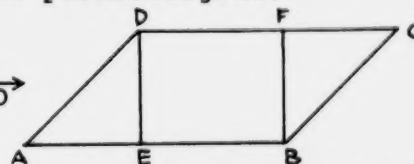
1. If the measure of one angle of a parallelogram is 45, find the measures of the other angles.
2. In $\square ABCD$, $m\angle A = 2x - 60$ and $m\angle D = x + 30$. Find the measures of the angles of this parallelogram.

3. Given: $\square ABCD$

$$\overleftrightarrow{DE} \perp \overleftrightarrow{AB} \text{ and } \overleftrightarrow{BF} \perp \overleftrightarrow{CD}$$

Prove: $\overline{DE} \cong \overline{BF}$

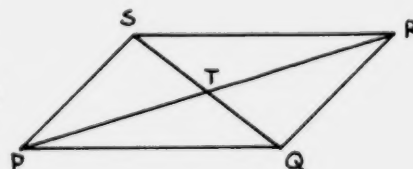
(Note: This example proves the theorem, "Parallel lines are (everywhere) equidistant.")



4. Given: $\square PQRS$

$$PT = 2x - 10, TR = x + 5$$

Find: PR



SUMMARY:

What are the properties of a parallelogram due to the

- definition of a parallelogram?
- theorems we have proved today?

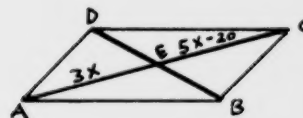
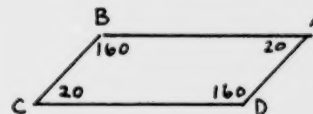
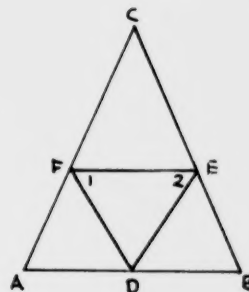
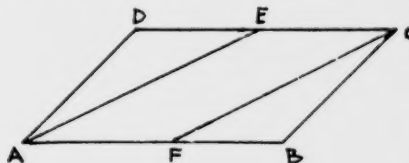
HOMEWORK:

- In $\square ABCD$, $m\angle C = 20$. Find the remaining angle measures.
- In $\square ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. If $AE = 3x$ and $EC = 5x - 20$, find AC.

- Given: $\square ABCD$
 $\overrightarrow{AE}$ bisects $\angle DAB$
 $\overrightarrow{CF}$ bisects $\angle BCD$

Prove: $\overline{AE} \cong \overline{CF}$

- Given: $\square ADEF$, $\square DBEF$
 $\overline{DE} \cong \overline{DF}$
 Prove: $\triangle ABC$ is isosceles

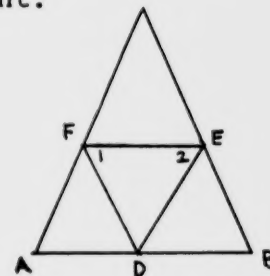


HOMEWORK SOLUTIONS:

- $m\angle A = 20$
 $m\angle B = 160$
 $m\angle D = 160$
- $3x = 5x - 20$
 $x = 10$

Thus, $AE = EC = 30$ and $AC = 60$.

- Prove $\triangle ADE \cong \triangle CBF$ by ASA $\cong$ ASA and use corresponding parts of congruent triangles are congruent.
- Since $\overline{DE} \cong \overline{DF}$, $\angle 1 \cong \angle 2$.
 From the given $\square$'s, $\angle 1 = \angle B$ and $\angle 2 = \angle A$. Thus, $\angle A \cong \angle B$ and $\overline{AC} \cong \overline{BC}$.



Lesson 46 - Quadrilaterals

AIM: How do we prove that a quadrilateral is a parallelogram?

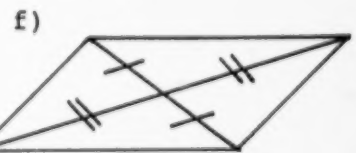
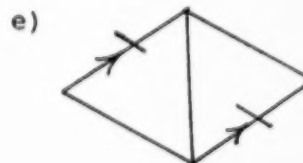
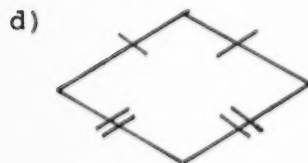
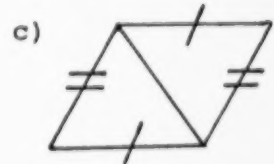
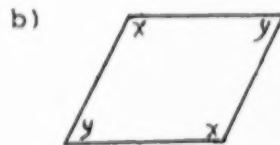
PERFORMANCE OBJECTIVES: The student will be able to...

1. State five methods of proving a quadrilateral is a parallelogram:
 - a) opposite sides parallel (both pairs)
 - b) opposite sides congruent (both pairs)
 - c) both pairs of opposite angles congruent
 - d) diagonals bisect each other
 - e) one pair of opposite sides both congruent and parallel
2. Apply these methods in numerical and algebraic problems.
3. Write original proofs that a quadrilateral is a parallelogram.

VOCABULARY: parallelogram, quadrilateral

MOTIVATION:

1. What is the definition of a parallelogram?
2. In each case, tell whether the marked information leads us to information sufficient for a quadrilateral to be a parallelogram.



3. Show in above:
 - a) meets the requirements of the definition
 - b) $2x + 2y = 360$, so $x + y = 180$ giving two pairs of

opposite sides parallel

- c) Prove the triangles congruent and get alternate interior angles congruent, leading to two pairs of opposite sides parallel.
- d) The opposite sides are NOT forced to be parallel.
- e) Prove the two triangles congruent and get the second pair of opposite sides parallel.
- f) Prove the triangles congruent (two pairs of congruent triangles), yielding two pairs of opposite sides parallel.

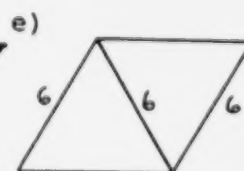
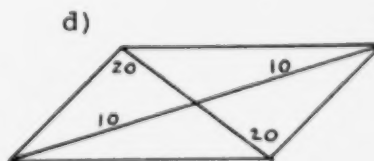
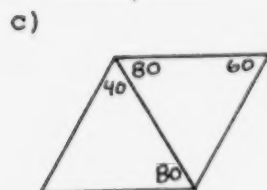
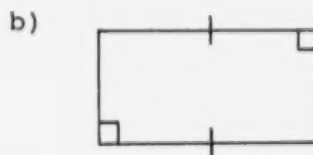
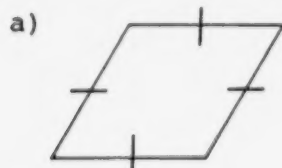
DEVELOPMENT:

1. In our previous problems five of the situations yielded parallelograms. We use these as different ways of proving a quadrilateral is a parallelogram.

Theorems:

If both pairs of opposite sides are parallel
OR both pairs of opposite sides are congruent
OR both pairs of opposite angles are congruent
OR one pair of opposite sides is parallel and congruent
OR the diagonals bisect each other
Then the quadrilateral is a parallelogram.

2. In each case, state why or why not the quadrilateral is known to be a parallelogram.

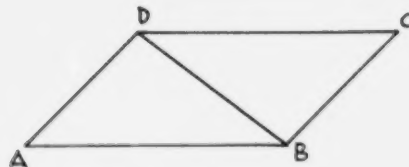


APPLICATIONS:

1. Given: $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$

$$\angle A = \angle C$$

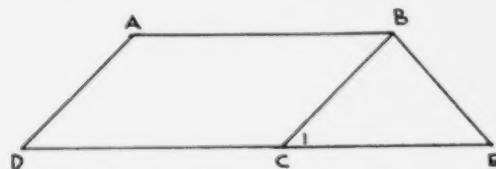
Prove: ABCD is a $\square$ P



2. Given: $\overline{AB} = \overline{CD}$, $\overline{AD} = \overline{BE}$

$$\angle 1 = \angle E$$

Prove: ABCD is a $\square$ P



SUMMARY:

When is a quadrilateral a parallelogram?

HOMEWORK:

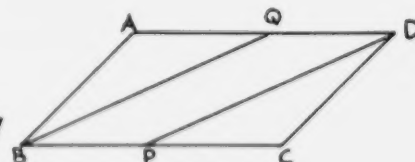
1. Tell whether each condition is sufficient for a quadrilateral to be a parallelogram. Draw a diagram in each case to use as part of your explanation.

- All four sides are congruent.
- Two pairs of angles are congruent.
- The consecutive angles are supplementary.
- Two sides are congruent.
- The diagonals are perpendicular to each other.

2. Given: $\square$ P ABCD

Q and P are the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively.

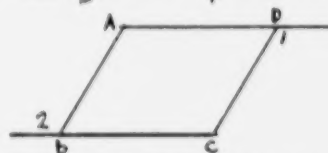
Prove: Quadrilateral BPDQ is a $\square$ P



3. Given: $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$

$$\angle 1 \cong \angle 2$$

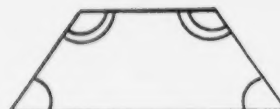
Prove: $\overline{AB} \cong \overline{CD}$



4. Prove that the quadrilateral formed by connecting the consecutive midpoints of the sides of a $\square$ P forms another $\square$ P.

HOMEWORK SOLUTIONS:

1. a) Yes.
 b) Not necessarily, as
 this diagram shows (an isosceles trapezoid).
 c) Yes, since opposite sides will be parallel.
 d) Not good enough. (See diagram b) above.)
 e) No (a "kite").



2. $\overleftrightarrow{QD}$ is parallel to $\overleftrightarrow{BP}$ from ABCD being a parallelogram.
 $\overline{QD} = \overline{BP}$ since halves of congruent segments are congruent.
 Thus, ABCD is a parallelogram.
3. Since $\overleftrightarrow{AD} // \overleftrightarrow{CB}$, $\angle A \cong \angle 2$.
 Since $\angle 1 = \angle 2$ also, $\angle A \cong \angle 1$ and $\overleftrightarrow{AB} // \overleftrightarrow{CD}$.
 Thus ABCD is a parallelogram and $\overline{AB} \cong \overline{CD}$.
4. Given: ABCD is a $\square$ with
 midpoints E, F, G, H of
 $\overline{DA}$, $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$
 respectively.

Prove: EFGH is a $\square$

One approach: First prove $\triangle DEH \cong \triangle BGF$ to get $\overline{EH} \cong \overline{GF}$.
 Then prove $\triangle AEF \cong \triangle CGH$ to get $\overline{EF} \cong \overline{GH}$.

Lesson 47 - Quadrilaterals

AIM: What are the properties of a rectangle?

How do we prove that a quadrilateral is a rectangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define rectangle.
2. State two properties.
 - a) A rectangle is equiangular.
 - b) The diagonals of a rectangle are congruent.
3. Solve simple algebraic and numerical problems involving rectangles.
4. Apply properties of a rectangle in proofs.
5. State methods of proving that a quadrilateral is a rectangle.
6. Write original proofs proving that a quadrilateral is a rectangle.

VOCABULARY: rectangle, equiangular

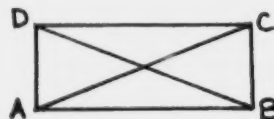
MOTIVATION:

1. In $\square ABCD$, $m\angle A = 90^\circ$.

Find $m\angle B$, $m\angle C$, $m\angle D$.

2. Given: $\square$ with $\angle DAB \cong \angle ABC$

Prove: $\overline{AC} = \overline{BD}$



3. What type of figure did we deal with in examples 1 and 2?
(Elicit: rectangles.) How would you define rectangle?
 - a) "A parallelogram with four right angles" is too much.
 - b) How many right angles are needed?

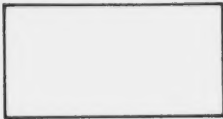
Definition: A RECTANGLE is a parallelogram with a right angle.

(Note: An alternative definition would be "a rectangle is an equiangular quadrilateral".)

DEVELOPMENT:

1. In our work today, we have already proved some properties of rectangles. Let's list these in a new page in our notebook. We will add other special quadrilaterals to this list later.

SPECIAL QUADRILATERALS

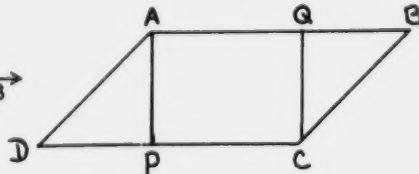
FIGURE	DEFINITION	THEOREMS
Rectangle 	A $\square P$ with a right $\angle$	1. A rectangle has ALL the properties of a $\square P$. 2. A rectangle has 4 right angles (is equiangular). 3. The diagonals of a rectangle are congruent.

2. Investigate:

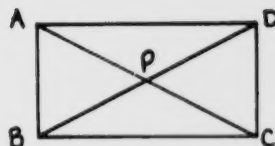
- a) Do the diagonals of a rectangle bisect the angles?
 - b) Must the diagonals of a rectangle be perpendicular to each other?
3. How can we prove that a quadrilateral is a rectangle?
 - a) From definition: A quadrilateral that is a parallelogram with a right angle is a rectangle.
 - b) An equiangular quadrilateral is a rectangle.
(Prove informally or alternate form of definition.)
 - c) If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle. (Prove informally by showing overlapping triangles congruent by SAS = SAS, giving congruent consecutive angles. Therefore, each is a right angle.)

APPLICATIONS:

1. In rectangle ABCD, $AC = 3x + 2$, $BD = 5x - 10$. Find x , AC and BD . (6, 20, 20)
2. Given: $\square P$ ABCD with
 $\overleftrightarrow{AP} \perp \overleftrightarrow{DC}$ and $\overleftrightarrow{CQ} \perp \overleftrightarrow{AB}$
 Prove: AQCP is a rectangle.



3. Given: ABCD is a rectangle
 Prove: $\triangle APD$ is isosceles



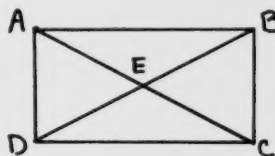
SUMMARY:

What are the major properties of a rectangle?

HOMEWORK:

1. In rectangle ABCD, $AC = 12$.

Find: a) BD b) BE c) AE



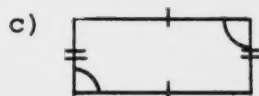
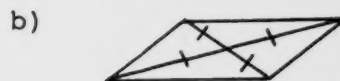
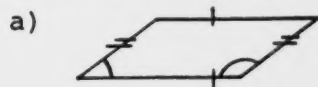
2. In rectangle ABCD, $AE = 5$.

Find BD.

ex. 1-2

3. In rectangle ABCD, $AC = 3x - 2$ and $BD = 5x - 12$. Find x, AC, BD.

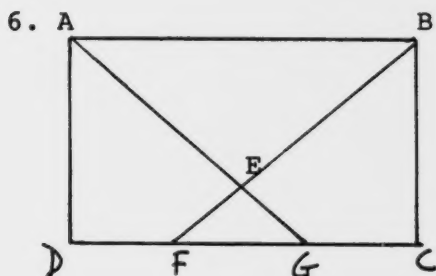
4. Which of the following MUST be a rectangle? Do not be misled by the diagram.



5. In $\square P$ ABCD, the diagonals meet at E. If $DE = 3x - y$, $EB = 2x + y$, $AE = 4x - 21$, $EB = 5y$.

a) Find x and y.

b) Prove ABCD is a rectangle.



Given: Rectangle ABCD with

$$\overline{DF} \cong \overline{CG}$$

Prove: a) $\overline{AG} \cong \overline{BF}$

b) $\triangle FGE$ is isosceles

HOMEWORK SOLUTIONS:

1. a) 12 b) 6 c) 6
2. 10
3. $x = 5$, $AC = BD = 13$
4. a) Rectangle. Each $\angle$ measures 90° since we have congruent consecutive $\angle$'s of a P.
- b) Yes, a rectangle. First show a pair of opposite triangles congruent by SAS = SAS. This results in a pair of opposite sides both parallel and congruent. Since the diagonals are congruent, the parallelogram is a rectangle.
- c) No.
- d) Rectangle.
5. a) $x = 14$, $y = 7$ b) Each diagonal measures 70
Thus we have a parallelogram with congruent diagonals,
making it a rectangle.
6. a) Prove $\triangle ADG \cong \triangle BCF$ by SAS $\cong$ SAS.
- b) Since $\angle BFG \cong \angle AGF$, $\overline{FE}$ must be $\cong \overline{GE}$.

Lesson 48 - Quadrilaterals

AIM: What are the properties of a rhombus?

How do we prove that a quadrilateral is a rhombus?

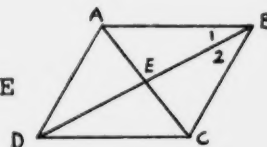
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define rhombus.
2. State four properties of a rhombus. It:
 - a) is equilateral.
 - b) has diagonals which are perpendicular to one another.
 - c) has diagonals which bisect the opposite angles.
 - d) has diagonals which form four congruent triangles.
3. Apply the properties of a rhombus to numerical and algebraic problems.
4. State four methods of proving that a quadrilateral is a rhombus.
5. Write original proofs proving that a quadrilateral is a rhombus.

VOCABULARY: rhombus

MOTIVATION:

1. In $\square ABCD$, $AB = 3$ and $BC = 3$. Find CD and DA .
2. Given: $\square ABCD$ with $\overline{AB} = \overline{CB}$
Prove:
 - a) $\triangle ABE \cong \triangle CBE \cong \triangle CDE \cong \triangle ADE$
 - b) $\angle 1 \cong \angle 2$
 - c) What other angles are congruent in this diagram?
3. Note in 2c) the four angles at E are congruent. Thus, each is a right angle.



4. What kind of figure did we deal with above?

Definition: A rhombus is a parallelogram which has a pair of congruent adjacent sides.

(Note: Alternately, we may define a rhombus as an equilateral quadrilateral.)

DEVELOPMENT:

1. In our work today, we have already proved some properties about rhombuses. Let's add these to the special page we started yesterday.

SPECIAL QUADRILATERALS

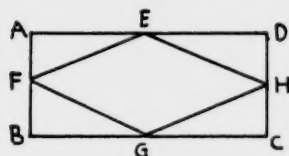
FIGURE	DEFINITION	THEOREMS
Rhombus 	A $\square$ with a pair of $\cong$ adjacent sides	1. A rhombus has all the properties of a $\square$. 2. A rhombus is equilateral. 3. Its diagonals a) form 4 $\cong$ triangles b) are $\perp$ to each other c) bisect the opposite $\angle$'s

2. Investigate: Are the diagonals of a rhombus congruent?
3. How can we prove that a quadrilateral is a rhombus?
 - a) From definition: A parallelogram with a pair of congruent adjacent sides is a rhombus.
 - b) An equilateral quadrilateral is a rhombus.
 - c) If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.
 - d) If a diagonal of a parallelogram bisects one of its angles, the parallelogram is a rhombus.
 (Prove b, c and d informally.)

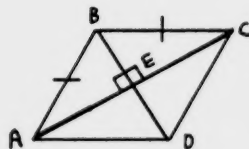
APPLICATIONS:

1. In rhombus ABCD, $AB = 5x - 2$, $CD = 3x + 10$. Find x and AB . (6,28)
2. In rhombus ABCD, $m\angle DAC = 3x - 10$ and $m\angle BAC = x + 20$. Find x and the two angle measures. (15,35,35)

3. Given: Rectangle ABCD with midpoints E, F, G and H of $\overline{DA}$, $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ respectively.
Prove: EFGH is a rhombus.



4. Given: Isosceles $\triangle ABC$,
 $\overline{BE} \perp \overline{AC}$, $\overline{AB} \cong \overline{CB}$, $\overline{BE} \cong \overline{ED}$
Prove: ABCD is a rhombus.

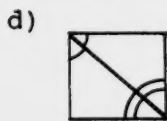
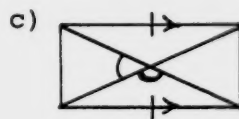
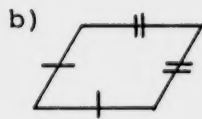
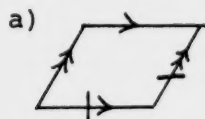


SUMMARY:

What are the major properties of a rhombus?

HOMEWORK:

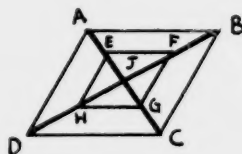
1. Which of the following MUST be rhombuses? Do not be misled by the diagram.



2. The diagonals of rhombus ABCD meet at E. Find:

- a) $m\angle ABD$ if $m\angle ADB = 20$
b) $m\angle AEB$

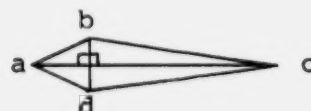
3. Given: Rhombus ABCD
E, F, G and H are the midpoints of $\overline{AJ}$, $\overline{BJ}$, $\overline{CJ}$ and $\overline{DJ}$ respectively.
Prove: EFGH is a rhombus.



4. If the diagonals of a quadrilateral are perpendicular, MUST the quadrilateral be a rhombus? Explain.
5. Prove that an equilateral quadrilateral is a rhombus.

HOMEWORK SOLUTIONS:

1.
 - a) Rhombus (by definition)
 - b) Not a rhombus
 - c) Rhombus (a parallelogram with perpendicular diagonals)
 - d) Not necessarily a rhombus
 - e) Does not have to be a rhombus. Not a rectangle unless diagonals also bisect each other or some other information is given.
2.
 - a) 20
 - b) 90
3. Since $\overline{AC}$ is perpendicular to $\overline{BD}$ and "halves of congruent segments are congruent", the four "little triangles" are congruent forming a parallelogram whose diagonals are perpendicular.
4. No, the perpendicular diagonals must also bisect each other.
5. First, show it is a parallelogram (both pairs of opposite sides are congruent). Now it is a parallelogram with a pair of congruent adjacent sides and so must be a rhombus.



Lesson 49 - Quadrilaterals

AIM: What are the properties of a square?

How do we prove that a quadrilateral is a square?

PERFORMANCE OBJECTIVES: The student will be able to...

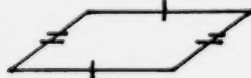
1. Define square (in two ways).
2. List the properties of a square.
3. Apply the properties in numerical and algebraic problems.
4. State two methods of proving that a quadrilateral is a square.
 - a) That it's a rectangle with two consecutive sides congruent.
 - b) That it's a rhombus with one right angle.
5. Write original proofs proving that a quadrilateral is a square.

VOCABULARY: square

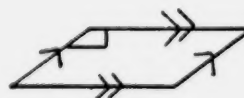
MOTIVATION:

1. In each case classify what special quadrilateral we have based on the information as marked. (Don't be misled by the given shape.)

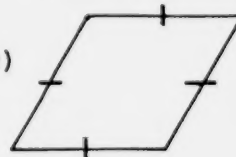
a)



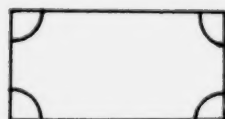
b)



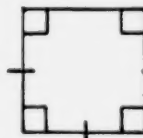
c)



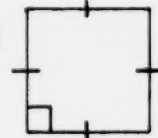
d)



e)



f)



- a) parallelogram, b) rectangle, c) rhombus, d) rectangle, e) and f) are squares
2. In e) above we have a SPECIAL RECTANGLE. We are given a pair of congruent consecutive sides but can show that all four sides are congruent. This is a square.
In f) above, we have a SPECIAL RHOMBUS. We are given one

right angle and can show that all of its angles are right angles.

3. In e) and f) show that a square is RHOMBUS and a RECTANGLE at the same time. Defining a square this way would give too much information. After all, would you define an isosceles triangle as, "A triangle with two sides congruent, two angles congruent, and whose altitude, median and angle bisector from the same vertex turn out to be the same line segment?"

DEVELOPMENT:

1. Elicit that e) and f) give hints as to how we can define a square without providing too much information. Hopefully, your students will give one of the following definitions, providing the MINIMUM amount of information needed.

Definition:

- a) A SQUARE is a rhombus with a right angle or
 - b) A SQUARE is a rectangle with a pair of congruent adjacent sides.
2. From either of these definitions, we can get all the properties of a square. Let's add this to our special page from earlier this week.

SPECIAL QUADRILATERALS

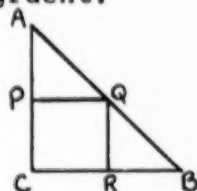
FIGURE	DEFINITION	THEOREMS
Square 	a) A rhombus with a right $\angle$ or b) A rectangle with a pair of $\cong$ adj. sides	1. A square has all the properties of a rectangle. 2. A square has all the properties of a rhombus. 3. The diagonals of a square cut the square into four $\cong$ isosceles right triangles.

3. Prove the third property on this list (informally).

APPLICATIONS:

- In square ABCD, diagonals $\overline{AC}$ and $\overline{BD}$ are drawn and intersect at E. Find
 - $m\angle BDC$
 - $m\angle BCE$
- How can we prove a quadrilateral is a square? Either
 - Use a previous method to prove it's a rhombus. Then show it has a right angle.
 - First show it's a rectangle. Then show that a pair of adjacent sides are congruent.

- Given: Isosceles right $\triangle ABC$ with right $\angle C$, $\overline{AC} \cong \overline{BC}$
 P, Q, R are the midpoints respectively of $\overline{AC}$, $\overline{AB}$, $\overline{BC}$



Prove:

- $\triangle APQ \cong \triangle QRB$
- PQRC is a rhombus
- PQRC is a square

SUMMARY:

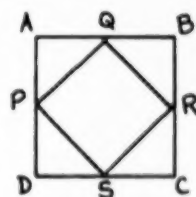
Go up and down the aisles of the class. Each student is to give you a different property of a square. (ex: Its diagonals bisect the opposite angles.)

HOMEWORK:

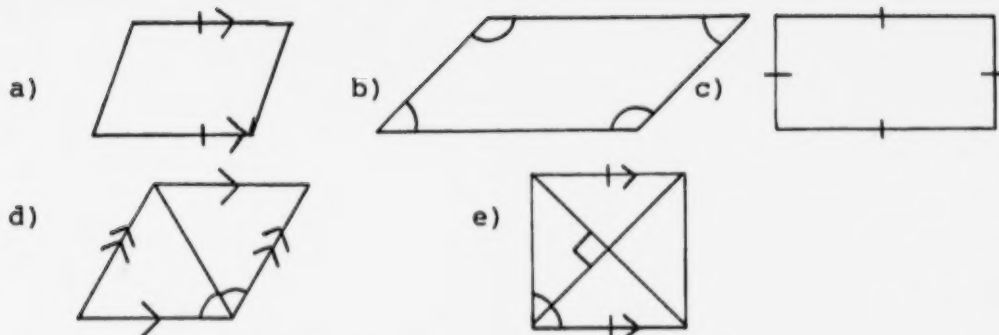
- Given: Square ABCD
 P, Q, R and S are the midpoints of $\overline{DA}$, $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ respectively.

Prove:

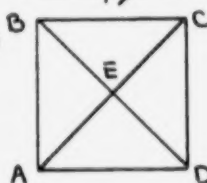
- $\triangle APQ \cong \triangle BQR \cong \triangle CRS \cong \triangle DSP$
- Find $m\angle DSP$, $m\angle RSC$
- Explain whether or not PQRS is also a square.



- In each case use the given information (as marked) to determine what special quadrilateral the figure MUST be. Aim for the MOST special figure and DO NOT be misled by the diagram.



3. Given: $\overline{AC}$ and $\overline{BD}$ bisect each other, $\overline{BD} \cong \overline{AC}$, $\overline{BD} \perp \overline{AC}$
 Prove: ABCD is a square.



HOMEWORK SOLUTIONS:

1.
 - a) Prove this using "halves of congruent segments are congruent" and SAS property.
 - b) 45, 45
 - c) Yes, since it is equilateral and from b) above, $m\angle PSR = 90$.
2.

a) parallelogram b) rectangle c) rhombus	d) rhombus e) a <u>rhombus</u> (unless an angle is a right angle or diagonals are $\cong$, in which case it is a square.)
--	---
3.
 - a) Since $\overline{AC}$ and $\overline{BD}$ bisect each other, ABCD is a parallelogram.
 - b) Since $\overline{BD} \cong \overline{AC}$, ABCD is a rectangle.
 - c) Since the diagonals are perpendicular, ABCD is also a rhombus. Thus, ABCD is a square.

Lesson 50 - Quadrilaterals

AIM: What are the properties of an isosceles trapezoid?
How do we prove that a quadrilateral is an isosceles trapezoid?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define trapezoid, isosceles trapezoid.
2. State and prove two properties of isosceles trapezoids:
 - a) diagonals are congruent
 - b) base angles are congruent
3. Do numerical and algebraic problems related to trapezoids.

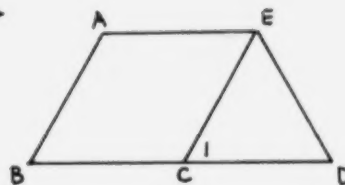
VOCABULARY: trapezoid, isosceles trapezoid, legs, bases, base angle

MOTIVATION:

1. Given: $\overleftrightarrow{AE} \parallel \overleftrightarrow{BD}$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{EC}$
 $\overline{AB} = \overline{ED}$

Prove:

- a) ABCE is a $\square$ P
- b) $\overline{EC} \cong \overline{ED}$
- c) Name ALL angles that MUST be congruent to $\angle D$
($\angle 1 \cong \angle D$, $\angle AEC \cong \angle D$, $\angle B \cong \angle D$)



2. Ask, "In this diagram, is ABDE a parallelogram? Why not?"

The above is a special quadrilateral that appears quite often. Therefore, we will need a special name for it.

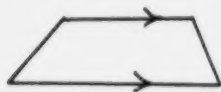
Definitions:

- 1) A **TRAPEZOID** is a quadrilateral with exactly two sides parallel.
- 2) The parallel sides of a trapezoid are called the **BASES**.
- 3) The non-parallel sides of a trapezoid are called the **LEGS**.

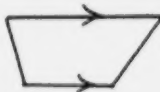
4) An ISOSCELES TRAPEZOID is a trapezoid with congruent legs.

3. Show several orientations for trapezoids.

a)



b)



c)



d)



DEVELOPMENT:

1. In our previous problem, we proved a theorem concerning isosceles trapezoids. Can you state it in words?

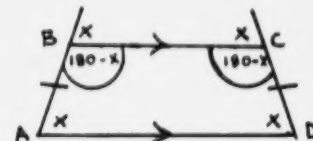
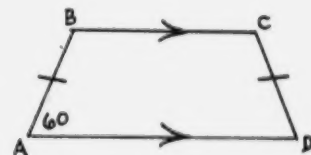
a) First define "base angles" as two angles of a trapezoid sharing a common base.

b) We have proved that the "lower base angles" of an isosceles trapezoid are congruent.

c) Find the remaining angle measures of this trapezoid. (Hint: extend $\overrightarrow{AB}$ and $\overrightarrow{DC}$)

$$(m\angle B = m\angle C = 120, m\angle D = 60)$$

d) In general, this diagram demonstrates that the "upper base angles of an isosceles trapezoid are congruent."



Thus we have **Theorem:** Base angles of an isosceles trapezoid are congruent.

2. a) Have the class draw four trapezoids - two isosceles, two non-isosceles.

The diagonals are very interesting.

Have the class draw them and conclude that "the diagonals of an isosceles trapezoid are congruent."

b) Prove by SAS = SAS

c) Ask, "In what other quadrilaterals must the diagonals be congruent?"



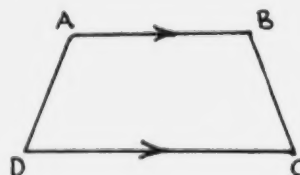
3. Add these results to our special page from earlier this week:

SPECIAL QUADRILATERALS

FIGURE	DEFINITION	THEOREMS
Isosceles Trapezoid	A trapezoid (quad. with 2 // sides) with $\cong$ legs	1. Base angles are congruent 2. Diagonals are congruent

APPLICATIONS:

- If the base angles of a trapezoid are congruent, MUST the trapezoid be isosceles? (Discuss with a good class. You can prove this by extending the legs until they intersect.) This is the converse of the motivation exercise.
- If $m\angle D = 30$, $m\angle B = 150$, (ex. 2-4)
 $AD = 2x + 5$ and $BC = x + 8$,
 find AD and BC . (11,11)
- If $\overline{AD} \cong \overline{BC}$, $m\angle D = 3x - 10$
 and $m\angle C = 2x + 5$, find $m\angle D$,
 $m\angle C$. (35,35)
- If $\overline{AD} \cong \overline{BC}$, $m\angle A = 3x + 10$ and $m\angle C = 2x + 20$, find x .
 (30)



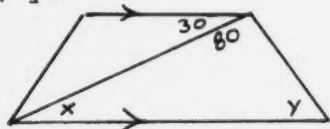
SUMMARY:

- What is an isosceles trapezoid?
- What are the special properties of an isosceles trapezoid?

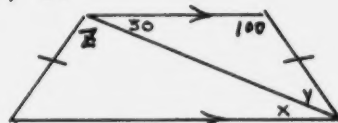
HOMEWORK:

1. Name the special quadrilaterals described:
 - a) It's a parallelogram with congruent diagonals.
 - b) It's a parallelogram with perpendicular diagonals.
 - c) It's not a parallelogram but it has two pairs of congruent angles.
 - d) It has exactly one pair of sides parallel and the other pair congruent.
 - e) It has four congruent sides.

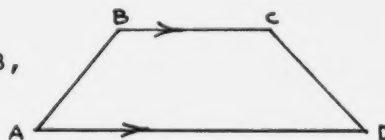
2. Find x , y .



3. Find x , y , z .

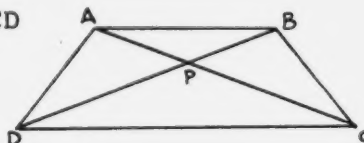


4. If $\angle B \cong \angle C$, $AB = 8x - 10$ and $CD = 3x + 5$, find x , AB , CD .



5. Determine whether or not an isosceles trapezoid can have a right angle as one of its base angles and explain your decision.

6. Given: Isosceles trapezoid ABCD with $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overline{AD} = \overline{BC}$.
Prove: $\triangle DPC$ is isosceles



HOMEWORK SOLUTIONS:

1.
 - a) rectangle
 - b) rhombus
 - c) isosceles trapezoid
 - d) isosceles trapezoid
 - e) rhombus
2. $x = 30$, $y = 70$
3. $x = 50$, $y = 30$, $z = 50$
4. $x = 3$, $AB = CD = 14$

5. Impossible - both base angles would be right angles as well as the other two angles and the figure would be a rectangle.
6. $\triangle ADC \cong \triangle BCD$ by SAS $\cong$ SAS
So $\angle PDC \cong \angle PCD \implies \overline{PD} \cong \overline{PC}$ so $\triangle DPC$ is isosceles.

Lesson 51 - Ratio and Proportion

AIM: What are ratios and proportions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define ratio, proportion, means, extremes, mean proportion, constant of proportionality.
2. State the theorem: In a proportion the product of the means equals the product of the extremes.
3. Determine if a proportion is valid.
4. Find the missing element of a proportion.
5. Find the mean proportional between two values.
6. Arrange four elements to form a valid proportion.
7. Solve simple ratio and proportion word problems.

VOCABULARY: ratio, proportion, means, extremes, mean proportional

MOTIVATION:

1. Prove: The line segment joining the midpoints of the legs of an isosceles triangle forms another isosceles triangle.
2. In the above problem, what is the relationship between the new and original triangle? (We use the term "similar" - note that the "ratio" between the corresponding sides of the two triangles is always the same, 1:2.)

Definition: A RATIO is the quotient of two quantities which are measured in the same units.

Examples:

- a) One measure is 3 in., a second is 1 ft. Their ratio is 3:12, $\frac{3}{12}$ or $\frac{1}{4}$. (Note - order is important.)

3. Often we state that two ratios are equal. Above: $\frac{3}{12} = \frac{1}{4}$.

Definition: A PROPORTION is an equation which states that two ratios are equal.

Examples:

1) $\frac{3}{4} = \frac{6}{8}$ 2) $1:2 = 4:8$ 3) $1:2 = 2:4$ 4) $5:10 = 10:20$

In geometry, two common types of proportions appear.

- a) $a:b = c:d$ (see ex. 1 and 2)
b) $a:b = b:c$ (see ex. 3 and 4)

DEVELOPMENT:

1. Definitions Involving Proportions:

In $a:b = c:d$ a is called the first proportional, b is the second proportional, etc.

a and d are called the EXTREMES

b and c are called the MEANS

In $a:b = b:c$ c is the third proportional

b is called the MEAN proportional

2. Find the fourth proportional to 3, 4, 5. ($\frac{3}{4} = \frac{5}{x} \rightarrow x = 6\frac{2}{3}$)
3. The last example illustrates how we've used proportions in the past.

Let's examine more closely: $\frac{3}{4} = \frac{6}{8}$ or $3:4 = 6:8$.

Theorem: In a proportion, the product of the means is equal to the product of the extremes.

4. Problems:

a) Solve for x : $12:8 = x:10$ (15)

b) Find the mean proportional between 4 and 7 ($\pm\sqrt{28}$)

Note: The positive result is called the "geometric mean".

5. Given $10 \times 6 = 5 \times 12$, can you invent a proportion that means the same as this equation? ($\frac{10}{5} = \frac{12}{6}$; $\frac{10}{12} = \frac{5}{6}$, etc.)

Theorem: If the product of two numbers is equal to the product of two other numbers, either pair may be made the means of a proportion, while the other pair is made the extremes. (If $ab = cd$, then $\frac{a}{c} = \frac{d}{b}$, etc.)

APPLICATIONS:

1. Solve $\frac{3}{x} = \frac{9}{27}$

2. Solve $\frac{8}{10} = \frac{12}{x-2}$

3. If $\frac{a}{b} = \frac{5}{7}$, find $\frac{b}{a}$.

4. If $\frac{a}{b} = \frac{2}{3}$, find $\frac{a+b}{b}$.

Show theorem: $\frac{a}{b} = \frac{c}{d} \rightarrow 1) \frac{a+b}{b} = \frac{c+d}{d} \quad 2) \frac{a-b}{b} = \frac{c-d}{d}$

5. Insert a mean proportional between

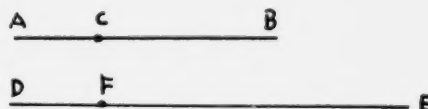
a) 4 and 9

c) 2 and 5

b) 16 and 4

6. Given: $\overline{AB}$ and $\overline{DE}$ are cut proportionally by C and F, respectively.

If $AC = 3$, $CB = 6$ and $DE = 15$, find DF and FE . (5, 10)



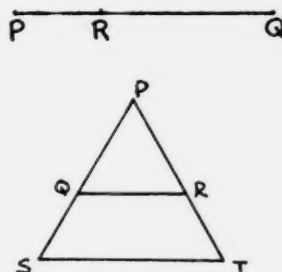
7. Divide 40 into the ratio 3:5.

SUMMARY:

What is the difference between a ratio and a proportion?

HOMEWORK:

- If $\frac{x}{3} = \frac{1}{2}$, find x .
- Insert a mean proportion between
 - 2 and 8
 - 7 and 10
- Form a proportion equivalent to $9 \times 4 = 12 \times 3$.
- Insert a mean proportion between $4\sqrt{2}$ and $2\sqrt{2}$.
- Test to see if this proportion is valid (or true).
 - $6:22 = 3:11$
 - $5:104 = 23:119$
- Divide \$36 into a ratio of 1:2.
- $\overline{PQ}$ has a length of 24 in.
Locate point R so that
 $PR:RQ = 1:3$
- Given: $\frac{PQ}{QS} = \frac{PR}{RT}$
 $PQ = 12$, $PS = 15$, $PR = 8$
Find: RT , PT

**HOMEWORK SOLUTIONS:**

- $x = 1.5$
- +4 or -4
 - $+\sqrt{70}$ or $-\sqrt{70}$
- $\frac{9}{12} = \frac{3}{4}$ or $\frac{9}{3} = \frac{12}{4}$ or $\frac{4}{12} = \frac{3}{9}$ or $\frac{4}{3} = \frac{12}{9}$
- 4 or -4
- Valid since $6 \times 11 = 22 \times 3$
 - Not valid since $5 \times 119 \neq 104 \times 23$
- \$12:\$24
- Choose R so that $PR = 6$.
- $\frac{12}{3} = \frac{8}{x}$
 $x = 2$
Thus $RT = 2$, $PT = 10$.

Lesson 52 - Ratio and Proportion - Similar Triangles

AIM: What are the proportions associated with a triangle?

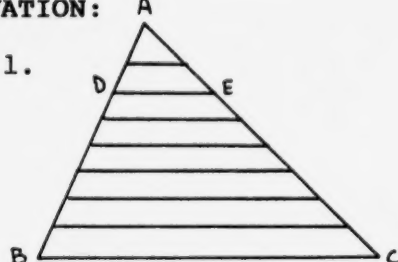
PERFORMANCE OBJECTIVES: The student will be able to...

1. State the meaning of "two segments are divided proportionally."
2. State the propositions:
 - a) If a line is parallel to one side of a triangle and intersects the other two sides, the line divides those segments proportionally (and its converse).
 - b) The segment joining the midpoints of two sides of a triangle is parallel to the third side and has length one-half the length of the third side.
3. Apply these propositions to numerical and algebraic problems.

VOCABULARY: commensurable, incommensurable, "two segments are divided proportionally"

MOTIVATION: A

1.



Given: In $\triangle ABC$, $\overline{AB}$ is divided into congruent segments, each with unit length by lines which are drawn parallel to $\overleftrightarrow{BC}$.

Find the ratio: $\frac{AE}{EC}$ (1:3)

2. The above problem leads to the following definitions.

Definitions:

- 1) A line segment is said to be COMMENSURABLE if it contains a rational number of unit lengths.
(ex. $8, 7, \frac{5}{3}$)
- 2) Otherwise, a line segment is said to be INCOMMENSURABLE. ($\sqrt{2}$, π , etc.)

3. The previous example is really the outline of the proof (commensurable case) of the "The Triangle Proportionality Theorem".

Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides these segments proportionally. (Discuss "divides these segments proportionally.")

Write (at least) eight resulting proportions.

1) $\frac{a}{b} = \frac{c}{d}$

2) $\frac{b}{a} = \frac{d}{c}$

3) $\frac{a}{c} = \frac{b}{d}$

4) $\frac{c}{a} = \frac{d}{b}$

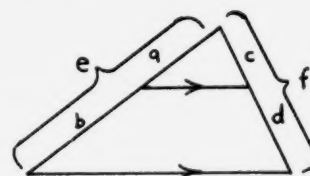
5) Add. gives $\frac{a+b}{b}$ or $\frac{c+d}{d}$ or $\frac{e}{b} = \frac{f}{d}$

6) $\frac{b}{e} = \frac{d}{f}$

7) $\frac{a}{e} = \frac{c}{f}$

8) $\frac{e}{a} = \frac{f}{c}$

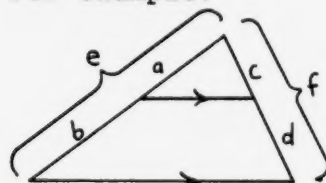
Also: $\frac{e}{f} = \frac{a}{c} = \frac{b}{d}$



DEVELOPMENT:

1. These proportions can be used to find missing lengths in a triangle when a parallel is drawn intersecting two sides.

For example:



- a) If $a = 8$, $b = 3$, $c = 16$, find d and f .

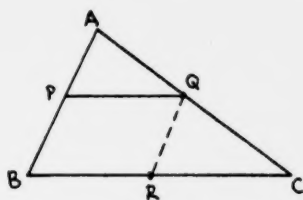
Elicit: $\frac{8}{16} = \frac{3}{x}$ (6,22)

- b) If $a = 6$, $b = 4$, $f = 20$, find d (8)

2. What is the converse of this theorem? Is it true? (yes)

Theorem: If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

3. This theorem has an interesting application:



Given: P is midpoint of $\overline{AB}$, Q is midpoint of $\overline{AC}$. What can we say about $\overleftrightarrow{PQ}$? ($\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$)

Now if we connect Q to R (where R is midpoint of $\overline{BC}$), we can say PQRB is a parallelogram. Why?

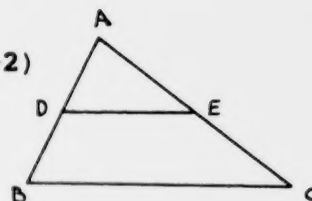
Thus since $BR = \frac{1}{2}BC$, $PQ = \frac{1}{2}BC$.

These results are often combined and worded as one theorem.

Theorem (corollary): The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.

APPLICATIONS:

- Given: D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{AC}$. $DE = 3x - 4$ and $BC = 4x + 12$. Find x and the two lengths. (x = 10, DE = 26, BC = 52)

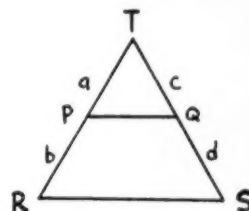


- If D and E are midpoints respectively of $\overline{AB}$ and $\overline{AC}$, and $AB = 8$, $AE = 9$, $BC = 12$, find DE.
- Prove that if successive midpoints of the sides of an equilateral triangle are connected, the resulting figure is another equilateral triangle.

SUMMARY:

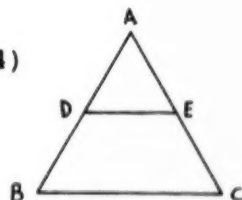
If $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$, name

- one resulting proportion
- one resulting product

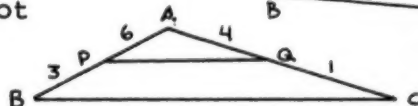
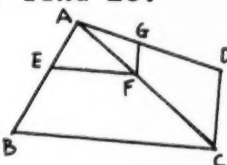


HOMEWORK:

- If D is the midpoint of $\overline{AB}$ (ex. 1-4)
and E is the midpoint of $\overline{AC}$
and $AB = 6$, $AC = 7$, $BC = 8$,
find DE.



- D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$, $DE = 2x - 4$, $BC = 2x + 14$, find x, DE, BC
- If $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $AD = 4$, $AB = 6$, $AE = 1$, find AC.
- If $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $AD = 2$, $AB = 3$, $AE = 4$, find EC.
- Given: $\overleftrightarrow{EF} \parallel \overleftrightarrow{BC}$ and $\overleftrightarrow{FG} \parallel \overleftrightarrow{CD}$
Prove: $\frac{AE}{AB} = \frac{AG}{AD}$
- Determine whether or not $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$. Explain.



HOMEWORK SOLUTIONS:

- $DE = 4$
- $x = 11$, $DE = 18$, $BC = 36$
- $\frac{4}{6} = \frac{1}{x} \rightarrow x = 1.5$
- $\frac{2}{3} = \frac{4}{x} \rightarrow x = 6$ so $EC = 2$
- $\frac{AE}{AB} = \frac{AF}{AC}$ since $\overleftrightarrow{EF} \parallel \overleftrightarrow{BC}$
 $\frac{AF}{AC} = \frac{AG}{AD}$ since $\overleftrightarrow{FG} \parallel \overleftrightarrow{CD}$
Thus, by transitive property, $\frac{AE}{AB} = \frac{AG}{AD}$
- $\overleftrightarrow{PQ}$ is not $\parallel \overleftrightarrow{BC}$ since $\overleftrightarrow{PQ}$ does not cut $\overline{AB}$ and $\overline{AC}$ proportionally. ($\frac{6}{3} \neq \frac{4}{1}$)

Lesson 53 - Similar Triangles

AIM: How do we prove triangles similar?

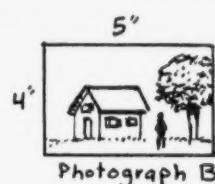
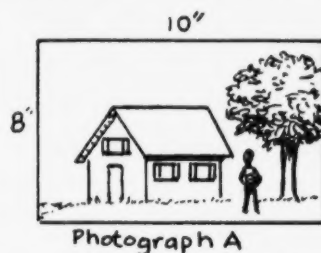
PERFORMANCE OBJECTIVES: The student will be able to...

1. Define similar polygons.
2. Show " $\sim$ " is an equivalence relation.
3. Distinguish between " $\sim$ " and " $\cong$ ".
4. State the theorem "Two triangles are similar if 2 angles of one triangle are congruent to 2 corresponding angles of the other triangle." ($aa = aa$) as an AXIOM.
5. Construct a triangle similar to given triangle on a given line segment as a base.
6. Apply the theorem in numerical exercises.

VOCABULARY: similar polygons, ratio of similitude

MOTIVATION:

1.



Photograph A (8" x 10") is an enlargement of photograph B (4" x 5") with no distortions or any pieces missing.

- a) If the side of the barn measures 4" x 3" in photograph A, what will its corresponding dimensions be in photograph B? ($2" \times 1\frac{1}{2}"$)
- b) If the stick figure (man) is $2\frac{1}{2}"$ high in photograph A, how high is it in photograph B? ($1\frac{1}{4}"$)
- c) If the top of the barn in photograph A is an isosceles triangle whose base angles measure 50, describe the top of the barn in photograph B.

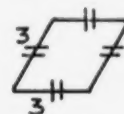
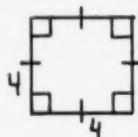
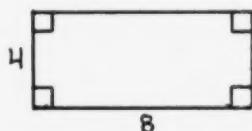
The above lead to:

Definitions:

- 1) Two polygons are SIMILAR if there is a one to one correspondence between the vertices such that
 - a) the corresponding angles are congruent and
 - b) the ratios of the measures of the corresponding sides are equal.
- 2) This common ratio is called the RATIO OF SIMILITUDE.

DEVELOPMENT:

1.



The above shows that

- a) Even though the rectangle and square have their "corresponding" angles congruent, they are not similar since their corresponding sides are not proportional.
- b) Even though the square and rhombus have their sides in proportion, they are NOT similar since their "corresponding" angles are not congruent.

BOTH CRITERIA MUST HOLD.

2. Show similarity is an Equivalence Relation.

- a) Reflexive Property: A polygon is similar to itself (ratio of similitude is 1:1).
- b) Symmetric Property: If figure I is similar to figure II, figure II is similar to figure I. (A "similarity" may be reversed.)
- c) Transitive Property: If polygon I is similar to polygon II and polygon II is similar to polygon III, polygon I is similar to polygon III. (Two figures similar to the same figure are similar to each other.)

3.



a) Given $\triangle ABC$ with $AC = 2$ inches and $DE = 1$ inch, can we construct a triangle similar to $\triangle ABC$ with DE as its base?

1) Duplicate $\angle A$ at D

2) Duplicate $\angle C$ at E

Connect so that a triangle is formed ($\triangle DEF$).

b) Now measure $\overline{DF}$ and $\overline{EF}$, $\overline{AB}$ and $\overline{CB}$. (Point out that $AB:DF = CB:EF = 2:1$, hopefully!)

It appears that the two triangles are similar. We write this as follows:

Axiom: If two angles of one triangle are congruent to two angles of another triangle, the triangles are similar. (aa = aa)

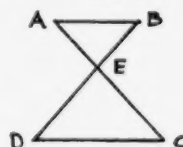
APPLICATIONS:

1. Are congruent triangles similar? Explain.

2. Are similar triangles congruent? Explain.

3. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Prove: $\triangle ABE \sim \triangle CDE$



4. In example 3, if $AB = 3$,
 $AE = 2$, $BE = 4$ and $DE = 8$,
find DC .

SUMMARY:

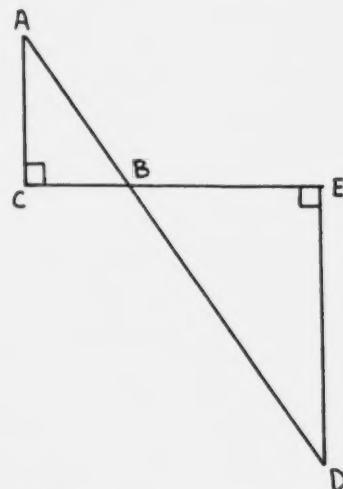
1. What are similar polygons?

2. How do we prove triangles similar?

HOMEWORK:

Which figures must be similar?

1. Two isosceles triangles
2. Two isosceles triangles with congruent vertex angles
3. Two right triangles
4. Two equilateral triangles
5. Two squares
6. Two rectangles
7. Two right isosceles triangles
8. Two rhombuses
9. A rhombus with congruent diagonals and a square
10. Two congruent triangles
11. Given: $\overleftrightarrow{AC} \perp \overleftrightarrow{CE}$, $\overleftrightarrow{DE} \perp \overleftrightarrow{CE}$
Prove: $\triangle ABC \sim \triangle DBE$



HOMEWORK SOLUTIONS:

- | | |
|--------|---------|
| 1. No | 6. No |
| 2. Yes | 7. Yes |
| 3. No | 8. No |
| 4. Yes | 9. Yes |
| 5. Yes | 10. Yes |
11. $\angle C \cong \angle E$ (all right angles are congruent)
 $\angle ABC \cong \angle DBE$ (vertical angles are congruent)
Thus, $\triangle ABC \sim \triangle DBE$ (aa $\cong$ aa).

Lesson 54 - Similar Triangles

AIM: How do we prove triangles similar? (Part II)

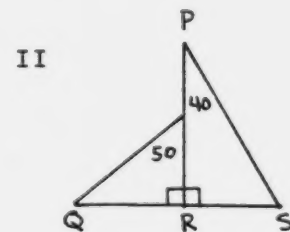
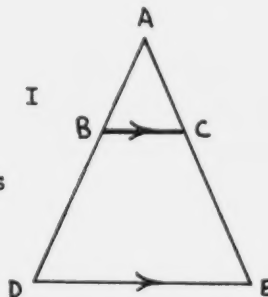
PERFORMANCE OBJECTIVES: The student will be able to...

1. State two corollaries to the $aa \cong aa$ theorem:
 - a) A line parallel to one side of a triangle and intersecting the other two sides cuts off a triangle similar to the original.
 - b) Two right triangles are similar if they agree in one acute angle.
2. State theorem: Two triangles are similar if their corresponding sides are in proportion.
3. Apply the methods of proving triangles similar in numerical problems and proofs.

VOCABULARY: no new terms

MOTIVATION:

1. In each case prove two triangles similar. Name the pairs of triangles you proved similar.



2. The above leads to the following corollaries:

Theorems (corollaries):

- 1) A line parallel to one side of a triangle and intersecting the other two sides cuts off a triangle similar to the original triangle (see I above).
- 2) Two right triangles are congruent if an acute angle of one triangle is congruent to an acute angle of the other.

DEVELOPMENT:

Although there are several methods of proving triangles similar, our major method will be $aa = aa$.

(Optional) We will investigate one other method.

1. Recall: For two polygons to be similar,
 - a) their corresponding angles had to be congruent and
 - b) their corresponding sides must be proportional.
2. For triangles, condition a) alone was sufficient. Let us see if condition b) alone is sufficient for two triangles to be similar.

Figure A seems to be an "enlargement" of figure B, where each side was enlarged by multiplying its length by 10 (as in our photograph

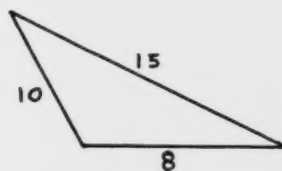


Figure A

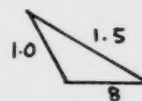


Figure B

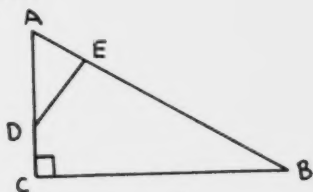
yesterday). This intuitively should have no effect on the angles. Although this can be proved, we will accept the following.

Axiom: If the lengths of the corresponding sides of two triangles are proportional, the two triangles are similar. ("SSS Similarity Theorem")

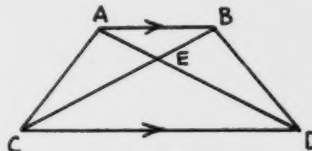
APPLICATIONS:

In each case (1-4) state which triangles are similar. Explain.

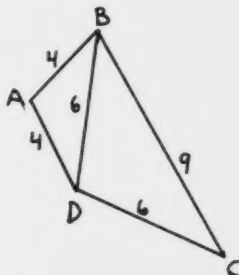
1.



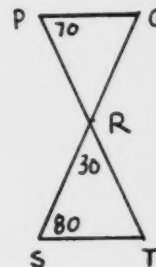
2.



3.



4.

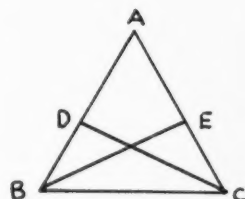


5. In $\triangle ABC$, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$, $\overleftrightarrow{PQ}$ intersects $\overline{AB}$ in P and $\overline{AC}$ in Q. If $PQ = 5$, $BC = 15$ and $AP = 3$, find AB.

6. Write out a proof:

Given: Isosceles $\triangle ABC$ with $\overline{AB} = \overline{AC}$, $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$

Prove: $\triangle BCD \sim \triangle CBE$



SUMMARY:

How can we prove two triangles are similar?

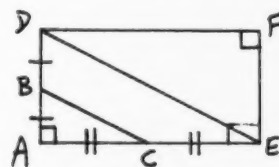
HOMEWORK:

In each case (1-4) state which two triangles are similar. Explain.

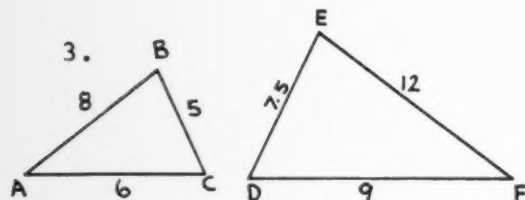
1.



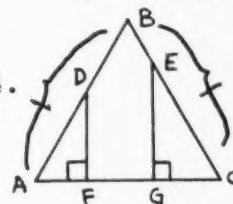
2.



3.



4.



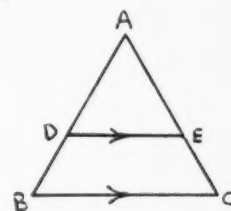
5. $AD = 3$, $DE = 4$, $BC = 12$, find AB.

6. $AE = 2$, $EC = 4$, $DE = 3$, find BC.

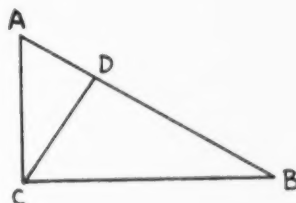
7. Given: In right $\triangle ABC$, $\angle ACB$ is a right angle, $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$

Prove:

- $\triangle ADC \sim \triangle ACB$
- $\triangle CDB \sim \triangle ACB$
- $\triangle ADC \sim \triangle CDB$



(ex. 5-6)



HOMEWORK SOLUTIONS:

1. $\triangle CDE \sim \triangle CAB$ by "line parallel to a side of a triangle cuts the triangle so that the new triangle is similar to the original."
2. $\triangle ABC \sim \triangle ADE$ (Since $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$, use above theorem.) In fact $\triangle ABC \sim \triangle ADE \sim \triangle FED$.
3. $\triangle ABC \sim \triangle FED$ (sides are in proportion)
4. $\triangle ADF \sim \triangle CEG$ (by aa = aa)
5. $\frac{3}{4} = \frac{x}{12} \longrightarrow x = 9$
6. $\frac{2}{6} = \frac{3}{x} \longrightarrow x = 9$
7. a) $\angle A \cong \angle A, \angle ADC \cong \angle ACB$
Thus $\triangle ADC \sim \triangle ACB$ by aa = aa.
b) $\angle B \cong \angle B, \angle CDB \cong \angle ACB$
Thus $\triangle CDB \sim \triangle ACB$ by aa = aa.
c) Combine a) and b) above, we get
 $\triangle ADC \sim \triangle CDB$ by the transitive property for similarity.

Lesson 55 - Similar Triangles

AIM: How do we prove that line segments are in proportion?

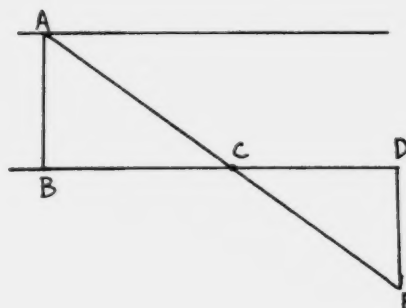
PERFORMANCE OBJECTIVES: The student will be able to...

1. Identify, given a proportion involving line segments, the triangles of which they are corresponding sides.
2. Write a proportion involving the corresponding sides of similar triangles.
3. Solve numerical and algebraic exercises involving proportions of similar triangles.
4. Write original proofs involving line segments that are in proportion and with a given segment as a mean proportional

VOCABULARY: similar triangles, corresponding segments, proportion, mean proportional

MOTIVATION:

AB is the distance across a river.
 $\overline{AB} \perp \overline{BD}$, $\overline{ED} \perp \overline{BD}$, $BC = 12$,
 $DC = 4$, $ED = 9$. Find the width (AB) of the river.



DEVELOPMENT:

1. $\frac{x}{9} = \frac{12}{4}$, $x = 27$
2. In order to write a proportion involving the corresponding sides of similar triangles, the corresponding angles which are congruent should be clearly labelled. The sides opposite the corresponding angles are the corresponding sides.

Given:

$$\text{Then } \frac{BC}{B'C'} = \frac{AC}{A'C'}$$

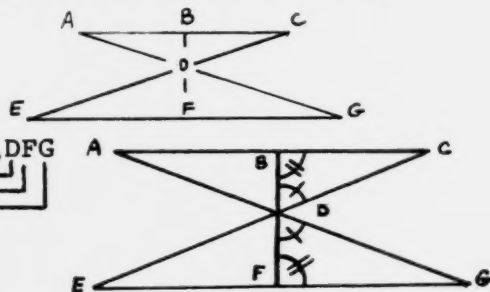


3. Sometimes, if there are more than two triangles in the diagram, it may be difficult to determine the corresponding sides. It may be helpful to draw the triangles separately.
4. If you are given a proportion involving line segments, then the angles opposite these segments should be labelled so that the similar triangles can be identified.

For example, given: $\frac{BD}{FD} = \frac{CD}{GD}$

Identify the angles as follows:

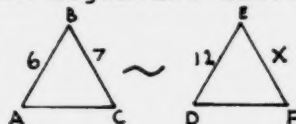
Similar triangles are: $\triangle DBC \sim \triangle DFG$



5. The following are numerical and algebraic exercises:

a) Given: $\triangle ABC \sim \triangle DEF$

Find x ($x = 14$)



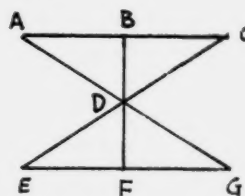
b) Given: $\angle C \cong \angle G$,

$\angle CBD \cong \angle GFD$

$CB = 3$, $GF = 4$, $CD = 7$

Find GD.

($\triangle CBD \sim \triangle GFD$, $\frac{CB}{GF} = \frac{CD}{GD}$, $GD = \frac{28}{3}$)



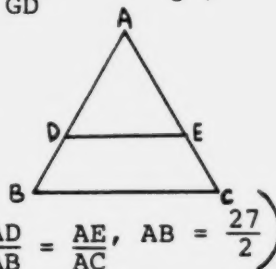
c) Given: $\overline{DE} \parallel \overline{BC}$,

(i) $AE = 8$, $AC = 12$,

$AD = 9$

Find AB.

($\triangle ADE \sim \triangle ABC$, $\frac{AD}{AB} = \frac{AE}{AC}$, $AB = \frac{27}{2}$)



(ii) Given: $AE = 4$, $AC = 6$, $AD = 5$

Find DB.

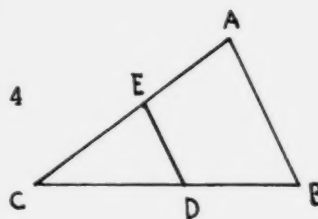
($DB = AB - AD$, $DB = 2\frac{1}{2}$)

d) Given: $\overline{ED} \parallel \overline{AB}$, $CD = 4$,

$DB = x - 2$, $CE = x$, $EA = x + 4$

Find CE.

($\triangle CED \sim \triangle CAB$, $\frac{CD}{CB} = \frac{CE}{CA}$)



$$\frac{4}{x+2} = \frac{x}{2x+4},$$

$$x(x+2) = 4(2x+4)$$

$$x^2 + 2x = 8x + 16, \quad x^2 - 6x - 16 = 0$$

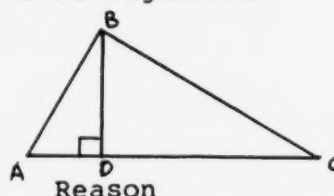
$$(x-8)(x+2) = 0, \quad x = 8$$

Note that students may need to solve quadratic equations by factoring in order to solve algebraic problems.

6. The following proof involves line segments.

Given: $\angle A \cong \angle CBD$,
 $\overline{BD} \perp \overline{AC}$

Prove: $\frac{AD}{BD} = \frac{BD}{CD}$



Statement

1. $\angle A \cong \angle CBD$, $\overline{BD} \perp \overline{AC}$

2. $\angle ADB$, $\angle BDC$ are right angles.

3. $\angle ADB \cong \angle BDC$

4. $\triangle ADB \sim \triangle BDC$

5. $\frac{AD}{BD} = \frac{BD}{CD}$

Reason

1. given

2. If 2 line segments are $\perp$, they form right angles.

3. All right angles are $\cong$.

4. AA similarity theorem

5. If 2 triangles are similar, then their corresponding sides are in proportion.

Note that $\overline{BD}$ is the mean proportion between $\overline{AD}$ and $\overline{CD}$.

APPLICATIONS:

Apply the algebraic exercises and proofs about triangles to triangles in quadrilaterals. Given: parallelogram ABCD, midpoints E, F, G and H. BE = 6, EF = 8, EH = 9.

1. Find AC and BD. (Draw $\overline{AC}$, $\overline{BD}$)

$$\triangle BEF \sim \triangle BAC, \triangle EAH \sim \triangle BAD, AC = 16, BD = 18$$

2. Prove EFGH is a parallelogram. (Find EF, FG, GH and HE. EFGH is a parallelogram since the opposite sides are congruent.)

SUMMARY:

1. How do you set up a proportion involving the line segments of similar triangles?
2. If you are given a proportion involving line segments, how can you identify the similar triangles?

HOMEWORK:

1. Given: $\triangle ABC \sim \triangle DEF$, $AB = 7$, $BC = 9$, $DE = 10$. Find: EF .

2. Given: $\triangle ABC$, D is a point of $\overline{AB}$, E is a point of $\overline{AC}$ $\overline{DE} \parallel \overline{BC}$. $AE = 8$, $AC = 12$, $AD = 10$. Find: AB .

3. Given: $\overline{DE} \parallel \overline{BC}$

a) Given: $AD = 4$, $DB = x - 4$,

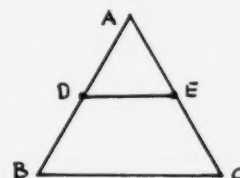
$AE = x$, $EC = x + 5$

Find: AE

b) Given: $AD = x + 3$, $DB = x + 6$, $AE = x + 2$,

$EC = x + 4$

Find: AC



4. Given: right angle BDC ,
right angle ABC

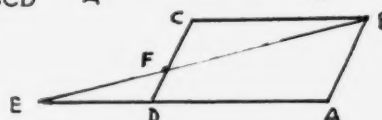
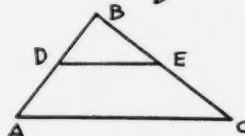
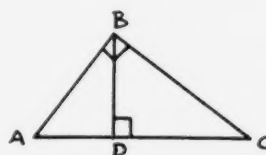
Prove: $\frac{AC}{AB} = \frac{AB}{AD}$

5. Given: $\overline{DE} \parallel \overline{BC}$

Prove: $\frac{AB}{DB} = \frac{BC}{BE}$

6. Given: parallelogram $ABCD$

Prove: $\frac{BF}{EF} = \frac{CF}{DF}$



HOMEWORK SOLUTIONS:

$$1. \frac{7}{10} = \frac{9}{x}$$

$$7x = 90$$

$$x = \frac{90}{7}$$

$$3a. \frac{4}{x} = \frac{x-4}{x+5}$$

$$x^2 = 8x + 20$$

$$x^2 - 8x - 20 = 0$$

$$(x - 10)(x + 2) = 0$$

$$x = 10$$

$$2. \frac{8}{12} = \frac{10}{AB}$$

$$8AB = 120$$

$$AB = 15$$

$$b. \frac{x+3}{2x+9} = \frac{x+2}{2x+6}$$

$$2x^2 + 12x + 18 = 2x^2 + 13x + 18$$

$$x = 0$$

$$AC = 2x + 6$$

$$AC = 6$$

4. Prove $\triangle ABC \sim \triangle BDA$ (by AA theorem of similarity)

$$5. \frac{AB}{AD} = \frac{BC}{EC} \implies \frac{AB}{AB-AD} = \frac{BC}{BC-EC}$$

6. Prove $\triangle BCF \sim \triangle EDF$ (by AA theorem of similarity)

Lesson 56 - Similarity Unit

AIM: How do we prove that products of the lengths line segments are equal?

PERFORMANCE OBJECTIVES: The student will be able to...

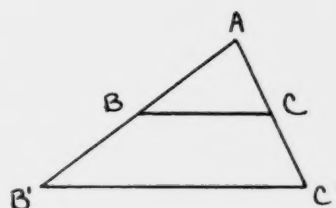
1. Form a product of line segments from a given proportion.
2. Form a proportion from a given product of line segments.
3. Develop a plan for a proof involving products of line segments in which
 - a) the product is converted to a proportion.
 - b) the triangles which correspond to this proportion are identified.
4. Solve numerical and algebraic exercises involving computation of lengths of sides of similar triangles, including indirect measurement and scale drawings.

VOCABULARY: similar triangles, proportion, product of line segments, scale, radical

MOTIVATION:

Consider a map with towns A, B and C. The map distance between towns A and B is 6 inches, while the map distance between A and C is 4 inches. The scale of the map is changed. B is

now named B' and C is named C'. The map distance between A and B' is now 9 inches. What is the map distance now between A and C'? (6 inches)



DEVELOPMENT:

1. If $\frac{a}{b} = \frac{c}{d}$, then the product of the means (b,c) equals the product of the extremes (a,d), $bc = ad$. Suppose that $\triangle ABC \sim \triangle A'B'C'$. Then $\frac{AB}{A'B'} = \frac{BC}{B'C'}$. Therefore, $AB \times B'C' = A'B' \times BC$.

2. If you are given a product, such as $ad = bc$, then by dividing both sides of the equation by bd (assume $bd \neq 0$), $\frac{ad}{bd} = \frac{bc}{bd}$ or $\frac{a}{b} = \frac{c}{d}$. Applying this to line segments, if $DE \times E'F' = D'E' \times EF$, then $\frac{DE}{D'E'} = \frac{EF}{E'F'}$.

3. A plan to prove that products of line segments are equal is:

- change the product into a proportion (see step 2)
- from the proportion, identify the triangles that need to be proven similar
- prove the corresponding triangles similar

4. Examples of proofs are:

a) Given:

$$\overline{AB} \cong \overline{AC}, \overline{AD} \perp \overline{BC}, \overline{CE} \perp \overline{AB}$$

$$\text{Prove: } AC \times EB = CB \times DC$$

(Show $\triangle ADC \sim \triangle CEB$)

b) Given: $\overline{AB} \parallel \overline{DE}$

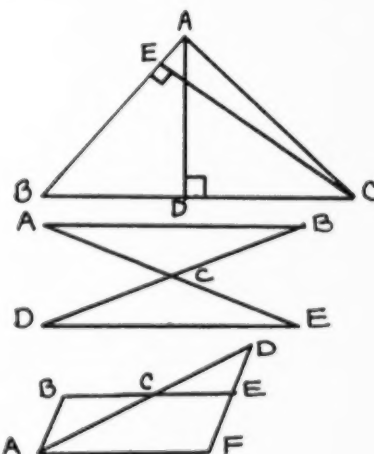
$$\text{Prove: } BC \times EC = DC \times AC$$

(Show $\triangle ABC \sim \triangle EDC$)

c) Given: parallelogram ABEF

$$\text{Prove: } AB \times EC = DE \times BC$$

(Show $\triangle ABC \sim \triangle DEC$)



5. The following are numerical and algebraic exercises:

a) Given: $\overline{DE} \parallel \overline{AC}$, $BD = 4$,

$$BA = 10, BC = 20$$

Find: BE (8)

b) Given: right $\triangle ACB$,

$$\overline{CD} \perp \overline{AB}, AB = 8, AC = 4$$

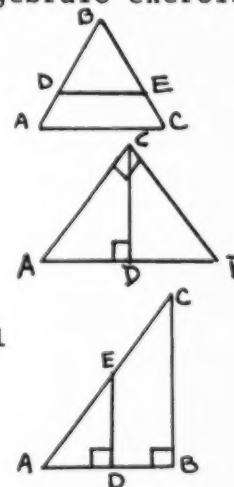
Find: AD (2)

c) Given: $\overline{CB} \perp \overline{AB}$, $\overline{ED} \perp \overline{AB}$,

$$CB = 3x, ED = x, AD = x + 1$$

$$AB = 4x + 1$$

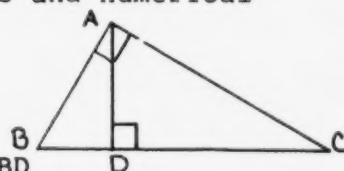
Find: x



APPLICATIONS:

Apply the product of line segments to proofs and numerical exercises involving right triangles:

Given: right $\triangle BAC$,
right $\triangle ADC$



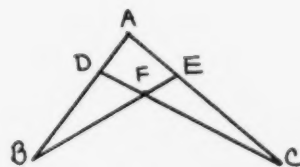
- 1) Prove: $AD \cdot AD = BD \cdot DC$, $AB \cdot AB = BC \cdot BD$
- 2) Find: AD if $BD = 6$, $DC = 5$ ($AD = \sqrt{30}$)
- 3) Find AB if $BD = 3$, $DC = 4$ ($AB = \sqrt{21}$)
- 4) Find x if $AB = x$, $BC = x + 3$, $BD = x - 2$ ($x = 6$)

SUMMARY:

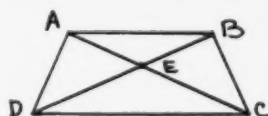
1. How do we find a product of line segments from a given proportion?
2. How do we form a proportion from a given product of line segments?
3. What is a plan for proofs requiring proving the products of line segments are equal?

HOMEWORK:

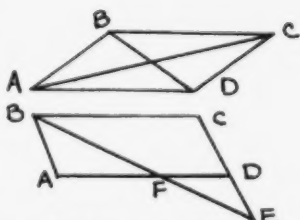
1. Given: $\triangle CDB \cong \triangle BEC$
Prove: $BF \times EF = CF \times DF$



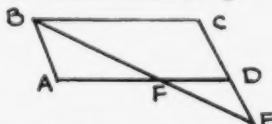
2. Given: trapezoid ABCD
 $\overline{AB} \parallel \overline{DC}$
Prove: $BE \times CD = AB \times DE$



3. Given: $\overline{AC}$ bisects $\angle BAD$,
 $\angle CBA \cong \angle ADC$
Prove: $AB \times DC = AD \times BC$

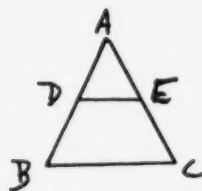


4. Given: parallelogram ABCD
Prove: $AF \times EF = DF \times BF$



- *5. In right triangle ABC, $\overline{CD}$ is the altitude drawn to hypotenuse $\overline{AB}$. The length of $\overline{DB}$ is 5 units longer than the length of $\overline{AD}$. If $CD = 3$, find the length of $\overline{AD}$ in radical form. (Note to teacher: quadratic formula needed to solve resulting equation.)

6. Given: $\overline{ED} \parallel \overline{CB}$, $AD = 6$,
 $DB = 4$, $AC = 15$
 Find: EC



HOMEWORK SOLUTIONS:

1. Show $\triangle BDF \sim \triangle CEF$
2. Show $\triangle ABE \sim \triangle CDE$
3. Show $\triangle ABC \sim \triangle ADC$
4. Show $\triangle ABF \sim \triangle DEF$
5. $\frac{AD}{3} = \frac{3}{AD+5}$ $AD = \frac{-5+\sqrt{61}}{2}$
6. let $x = EC$, $\frac{6}{4} = \frac{15-x}{x}$, $x = 6$

Lesson 57 - Similarity

AIM: What are the special proportions concerning corresponding line segments in similar polygons?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the following propositions concerning similar triangles:
 - a) lengths of corresponding altitudes have the same ratio as the lengths of any two corresponding sides.
 - b) lengths of two corresponding line segments have the same ratio as the lengths of any two corresponding sides.
 - c) perimeters of similar triangles have the same ratio as the lengths of any two corresponding sides.
2. State the equivalent propositions for similar polygons.
3. Apply these propositions in numerical and algebraic exercises with emphasis on indirect measurement and scale drawing.

VOCABULARY: altitude, polygon

MOTIVATION:

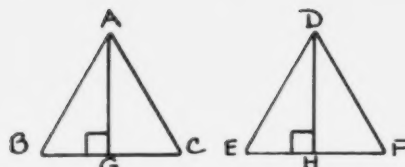
A rectangular photograph is enlarged so that the dimensions are doubled. If the perimeter of the original photograph is 24", what is the perimeter of the enlargement? (48")

DEVELOPMENT:

1. Using the definition of similar triangles, it follows that the lengths of corresponding altitudes (or any corresponding line segments) of similar triangles have the same ratio as the lengths of any two corresponding sides.

2. For example,

Given: $\triangle ABC \sim \triangle DEF$,
altitudes $\overline{AG}$ and $\overline{DH}$



Possible conclusions are: $\frac{AG}{DH} = \frac{AB}{DE}$, $\frac{BG}{EH} = \frac{AC}{DF}$, etc.

3. It also follows from the definition of similar triangles that the perimeters of similar triangles have the same ratio as the lengths of any two corresponding sides.

4. For example,

Given: $\triangle ABC \sim \triangle DEF$

A possible conclusion is: $\frac{AB+BC+CA}{DE+EF+FD} = \frac{AB}{DE}$

5. Since similar polygons may be divided into similar triangles by drawing diagonals between pairs of corresponding vertices, the following equivalent proposition may be stated for similar polygons: The lengths of corresponding altitudes (or corresponding line segments or corresponding perimeters) of similar polygons have the same ratios as the lengths of any two corresponding sides.

6. For example, Given: hexagon $ABCDEF \sim$ hexagon $A'B'C'D'E'F'$

A possible conclusion is:

$$\frac{AB+BC+CD+DE+EF+FA}{A'B'+B'C'+C'D'+D'E'+E'F'+F'A'} = \frac{AB}{A'B'}$$

7. The following are numerical and algebraic exercises:

- If the ratio of the corresponding sides of two similar triangles is 4:9, then what is the ratio of their perimeters? (4:9)
- Given: $\triangle ABC$ is similar to $\triangle DEF$. The perimeter of triangle ABC is 64, the perimeter of triangle DEF is 48. $AB = 18$. Find: DE . ($\frac{27}{2}$)
- The lengths of the sides of a triangle are 5, 7 and 8. If the length of the longest side of a similar triangle is 24, find the perimeter of the larger triangle. (60)

APPLICATIONS:

The lengths of any two corresponding medians (or angle bisectors) of two similar triangles have the same ratio as the lengths of any two corresponding sides.

Given: $\triangle ABC \sim \triangle DEF$

1. Given: medians $\overline{BG}$, $\overline{EH}$

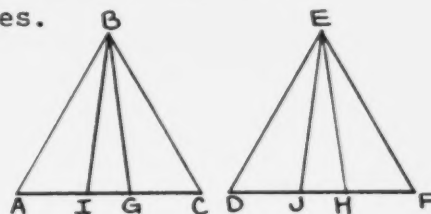
$$BG = 2EH, BC = 9$$

Find: EF (4.5)

2. Given: $\angle$ bisectors $\overline{BI}$, $\overline{EJ}$

$$BI = x, EJ = x + 3, AC = x - 2, DF = x$$

Find: x (6)



SUMMARY:

1. Which line segments in similar triangles have the ratio of their lengths equal to the ratio of the lengths of any two corresponding sides?
2. How does the ratio of perimeters of similar triangles compare to the ratio of the lengths of any two corresponding sides?

HOMEWORK:

1. The lengths of the sides of a triangle are 7, 9 and 11. If the length of the shortest side of a similar triangle is 35, find the perimeter of the larger triangle.
2. Given: $\triangle ABC \sim \triangle DEF$. The perimeter of $\triangle ABC$ is 72, the perimeter of $\triangle DEF$ is 54. $DE = 15$. Find: AB

3. Given: $\triangle ABC \sim \triangle DEF$

a) Given: medians $\overline{BG}$, $\overline{EH}$

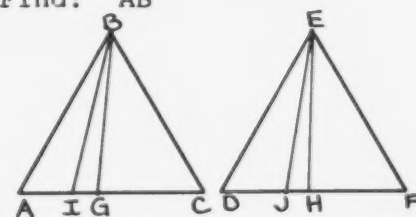
$$BG = 3EH, ED = 36$$

Find: BA

b) Given: $\angle$ bisectors $\overline{BI}$, $\overline{EJ}$

$$AB = x + 2, DE = x - 3, BI = x, EJ = x - 4$$

Find: x



HOMEWORK SOLUTIONS:

1. 135

2. $\frac{AB}{15} = \frac{72}{54}$, $AB = 20$

3. a) $BA = 108$

b) $\frac{x+2}{x-3} = \frac{x}{x-4}$

$$x^2 - 3x = x^2 - 2x - 8$$

$$x = 8$$

Lesson 58 - Similarity

AIM: How do we use proportions in the right triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a "projection of a point" and a "projection of a segment" on a line.
2. State the theorem, "If the altitude is drawn to the hypotenuse of a right triangle then:
 - a) the two triangles formed are similar to the original triangle and to each other;
 - b) the length of each leg of the original triangle is the mean proportional between the length of the hypotenuse and the length of the projection of that leg upon the hypotenuse;
 - c) the length of the altitude is the mean proportional between the lengths of the two segments of the hypotenuse."
3. Distinguish between rational and irrational numbers.
4. Solve numerical and algebraic exercises related to the theorem.

VOCABULARY: projection of a point, projection of a segment, mean proportional, rational numbers, irrational numbers

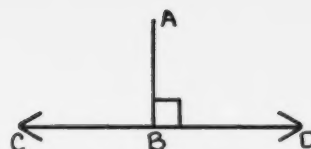
MOTIVATION:

The sides of a tent, $\overline{AB}$ and $\overline{CB}$, are perpendicular. The distances from the foot of the pole ($\overline{BD}$) are 4 (DA) and 9 (DC). What is the length of the pole $\overline{BD}$? (6)

DEVELOPMENT:

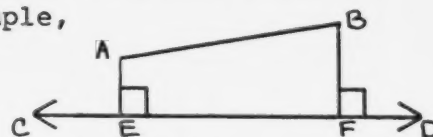
1. The projection of a point on a line is the foot of the perpendicular drawn from the point to the line. For example,

B is the projection of point A on $\overleftrightarrow{CD}$.



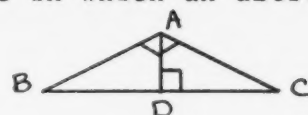
2. The projection of a given segment on a line is the line segment formed by the projection of each endpoint of the given segment on the line. For example,

$\overline{EF}$ is the projection of $\overline{AB}$ on $\overleftrightarrow{CD}$.



3. We now prove the theorem stated in the performance objective. Consider a right triangle in which an altitude is drawn to the hypotenuse:

Given: right angle BAC
altitude $\overline{AD}$

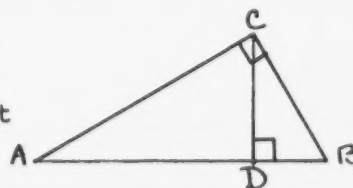


Prove:

- a) $\triangle BDA \sim \triangle ADC$, $\triangle BDA \sim \triangle BAC$, $\triangle ADC \sim \triangle BAC$
(Note that $\angle BAC \cong \angle ADB$, $\angle B \cong \angle CAD$, $\angle DAB \cong \angle C$. The triangles are proven similar by the AA theorem of similarity.)
- b) $\frac{BC}{BA} = \frac{BA}{BD}$ (BA is the mean proportional between the hypotenuse, BC, and the projection, BD, of $\overline{BA}$ on $\overline{BC}$.)
 $\frac{BC}{AC} = \frac{AC}{DC}$ ($\overline{AC}$ is the mean proportional between the hypotenuse, BC, and the projection, DC, of $\overline{AC}$ on $\overline{BC}$.)
(Use the definition of similar triangles to set up these proportions.)
- c) $\frac{BD}{AD} = \frac{AD}{CD}$ (AD is the mean proportional between the segments, BD and CD, of the hypotenuse.)
(Use the definition of similar triangles BDA and ADC to set up the proportion.)

4. Your answers to numerical and algebraic problems may be rational or irrational numbers. A rational number can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Examples of rational numbers are $\frac{3}{2}$, $-\frac{7}{9}$, $\frac{6}{1}$, 0. An irrational number cannot be expressed in the form of a rational number. Examples of irrational numbers are $\sqrt{3}$, $\sqrt{21}$, and π .

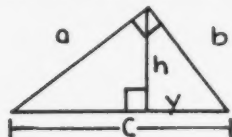
5. The following are numerical and algebraic exercises for the right triangle ACB with altitude $\overline{CD}$.



- a) Given: $AD = 5, BD = 7$
Find: CD ($\sqrt{35}$)
- b) Given: $AC = 9, AB = 7$
Find: AD ($AD = \frac{81}{7}$)
- c) Given: $AB = 11, AD = 4$
Find: CB ($DB = 7, CB = \sqrt{77}$)
- d) Given: $CD = 6, AD = x, BD = x + 5$
Find: AB ($\frac{x}{6} = \frac{6}{x+5}, x = 4, AB = 13$)
- e) Given: $CD = 12, AD$ exceeds BD by 7
Find: AD (let $BD = x, \frac{x}{12} = \frac{12}{x+7}, x = BD = 9$ and $AD = 9 + 7 = 16$)

APPLICATIONS:

Consider the following diagram, with right angles indicated.



Express all possible proportions involving the letters.

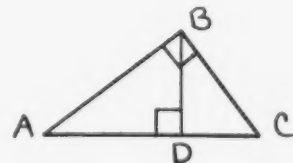
SUMMARY:

1. If an altitude is drawn to the hypotenuse of a right triangle, then:
 - a) what is the relationship of the two triangles formed and the original triangle?
 - b) state a proportion involving a leg of the original triangle.
 - c) state a proportion involving the altitude and the two segments of the hypotenuse.
2. What is a rational number?
3. Give examples of rational and irrational numbers.

HOMEWORK:

Given: right angle ABC ,
altitude $\overline{BD}$

1. Given: $AD = 7, CD = 6$
Find: BD
2. Given: $AD = 5, CD = 8$
Find: AB



*3. Given: $BD = x$, $AD = x + 1$, $CD = 5$

Find: x

4. Given: $AB = x$, $AD = x - 2$, $CD = 8$

Find: AB

*5. Given: $AB = x$, $AD = x - 3$, $CD = x + 2$

Find: x

6. Given: $BD = x$, $AD = x + 5$, $CD = x - 3$

Find: AC

(Note to teacher: quadratic formula needed to solve equations in ex. 3 and 5)

HOMEWORK SOLUTIONS:

1. $\sqrt{42}$

2. $\sqrt{65}$

3. $\frac{x+1}{x} = \frac{x}{5}$
 $x^2 - 5x - 5 = 0$
 $x = \frac{5 + \sqrt{45}}{2}$

4. $AC = x + 6$
 $\frac{x-2}{x} = \frac{x}{x+6}$
 $x^2 = x^2 + 4x - 12$
 $x = 3$

5. $AC = 2x - 1$
 $\frac{x-3}{x} = \frac{x}{2x-1}$
 $x^2 - 7x + 3 = 0$
 $x = \frac{7 + \sqrt{37}}{2}$

6. $\frac{x+5}{x} = \frac{x}{x-3}$
 $x^2 = x^2 + 2x - 15$
 $x = \frac{15}{2}$
 $AC = 2x + 2$
 $AC = 17$

Lesson 59 - Similarity

AIM: How do we prove and apply the Pythagorean theorem?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State and prove informally the Pythagorean theorem.
2. Define Pythagorean triples.
3. Solve numerical and algebraic exercises concerning sides of a right triangle by applying the Pythagorean theorem.
4. Find the square and square root of real numbers.
5. Express an irrational number in simplest radical form.
6. Apply the Pythagorean theorem in solving exercises such as computing the length of:
 - a) altitude of an equilateral or isosceles triangle
 - b) side of a rhombus given the diagonals
 - c) the altitude of an isosceles trapezoid
7. Form the converse of the Pythagorean theorem.
8. Determine if a set of 3 lengths could be for the sides of a right triangle.

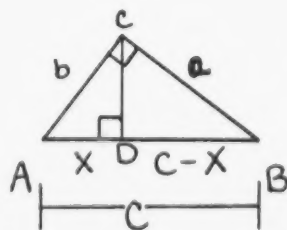
VOCABULARY: Pythagorean theorem, Pythagorean triples, radical form, rhombus, isosceles trapezoid, converse

MOTIVATION:

A doorway is 8' high and 6' wide. What is the widest rectangular panel that can be taken through the doorway? ("slightly" less than 10')

DEVELOPMENT:

1. In order to prove the Pythagorean theorem, consider the following right triangle ABC , with altitude $\overline{CD}$.



Prove: $a^2 + b^2 = c^2$ (In a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse.)

Use the theorem from the previous lesson for each leg:

$$\frac{c}{a} = \frac{a}{c-x}$$

$$a^2 = c^2 - cx$$

$$\frac{c}{b} = \frac{b}{x}$$

$$b^2 = cx$$

$$a^2 + b^2 = c^2 - cx + cx \text{ or}$$

$$a^2 + b^2 = c^2$$

2. A Pythagorean triple is a set of three integers (a,b,c) which could represent the lengths of the sides of a right triangle, that is, they satisfy the equation $a^2 + b^2 = c^2$. For example, the following are Pythagorean triples: (3,4,5), (5,12,13) and (8,15,17). Note that if each number in the Pythagorean triple is multiplied by a positive integer x, then another Pythagorean triple is formed (since the new numbers represent the sides of a similar right triangle). For example, (6,8,10) and (10,24,26) are Pythagorean triples.

3. Irrational numbers should be expressed in simplest radical form by factoring out any perfect squares. For example, $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$. To the nearest tenth, $3\sqrt{2}$ is 4.2.

4. The following are numerical and algebraic exercises (first express the answers in simplest radical form and then to the nearest tenth).

a) Given: $a = 24$, $b = 45$

Find: c

(The sides are three times the Pythagorean triple (8,15,17). $c = 51$)

b) Given: $a = 7$, $b = 9$

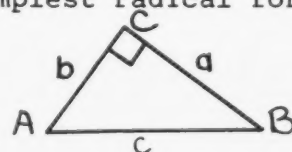
Find: c ($\sqrt{130}$ or 11.4 to the nearest tenth.

c) Given: $b = 4$, $c = 8$

Find: a ($4\sqrt{3}$ or 6.9 to the nearest tenth.

d) Given: $a = x$, $b = x + 2$, $c = x + 4$

Find: x (6)



5. The following are applications of the Pythagorean theorem:

a) Given: $AB = 9$, $AC = 9$

altitude $\overline{AD}$, $BC = 12$

Find: AD

($BD = 6$, $AD = \sqrt{45}$ or 6.7)

b) Given: rhombus $ABCD$

$BD = 30$, $AC = 16$

Find: BC (The diagonals

of a rhombus are perpendicular to each other.

$BE = 15$, $EC = 8$, $BC = 17$.)

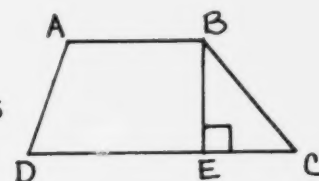
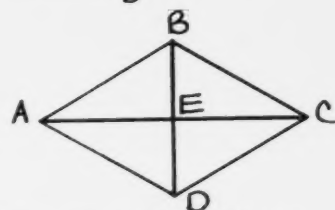
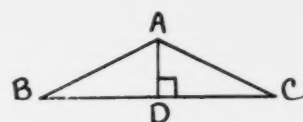
c) Given: isosceles trapezoid

$ABCD$, with $\overline{AB} \parallel \overline{CD}$, altitude $\overline{BE}$,

$AD = BC$, $AB = 12$, $DC = 20$, $BC = 5$

Find: BE (draw $\overline{AG} \perp \overline{DC}$, $GE = 12$,

$EC = 4$, $BE = 3$)



6. The converse of the Pythagorean theorem which is also a theorem is: If the sum of the squares of two sides of a triangle equals the square of the third side, then the triangle is a right triangle. Given the lengths of the three sides of a triangle, the converse can be used to determine if the triangle is a right triangle. For example, indicate which of the following triangles with the given side lengths are right triangles.

a) 7, 12, 14 (no, since $7^2 + 12^2 \neq 14^2$)

b) 40, 85, 75 (yes)

c) $\sqrt{5}$, 1, 2 (yes, since $(\sqrt{5})^2 = 1^2 + 2^2$)

d) $\sqrt{6}$, $\sqrt{23}$, 4 (no)

APPLICATIONS:

If you are asked to find the length of the side of any polygon and if there is a right angle in the diagram, then consider applying the Pythagorean theorem. For example,

1. In rectangle $ABCD$, $AB = x$, $BC = x + 7$, and diagonal $BD = x + 8$. Find BD . ($x^2 - 2x - 15 = 0$, $x = 5$, $BD = 13$)

2. The perimeter of a rectangle is 28 centimeters. If the length of a diagonal of the rectangle is 10 cm, find the number of centimeters in the length and width of the rectangle. (6,8)
3. What is the length of a diagonal of a square of side length 2. ($2\sqrt{2}$)

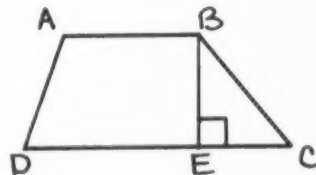
SUMMARY:

1. What is the Pythagorean theorem?
2. What is a Pythagorean triple?
3. State examples of Pythagorean triples.
4. How do you express an irrational number in simplest form?
5. What is the converse of the Pythagorean theorem?

HOMEWORK:

Express all answers first in simplest radical form and then to the nearest tenth.

1. The diagonals of a rhombus have lengths of 8 and 6. What is the length of a side of the rhombus?
2. In rectangle ABCD, $BC = 12$ and $CD = 16$. Find BD.
3. What is the perimeter of a square which has a diagonal of length 12?
- *4. In right triangle ABC, $BC = x$, $AC = 8 - x$, and hypotenuse $AB = 6$. Find BC. (Note to teacher: quadratic formula required to solve equation.)
5. Given: equilateral triangle ABC, altitude $\overline{AD}$, $AB = 14$. Find: AD
6. Given: isosceles trapezoid ABCD, $\overline{AB} \parallel \overline{DC}$, $AD = BC$, altitude $\overline{BE}$, $BE = 9$, $AB = 16$, $DC = 22$
Find: BC



7. Can a triangle with side lengths of $x + 1$, $5x$ and $5x + 1$ be a right triangle?

HOMEWORK SOLUTIONS:

1. 5
2. 20
3. $24\sqrt{2}$ or 33.9
4. $x^2 - 8x + 14 = 0$
 $x = \frac{8 - \sqrt{8}}{2}$
 $x = 4 - \sqrt{2}$ or 2.6
 $x = \frac{8 + \sqrt{8}}{2}$
 $x = 4 + \sqrt{2}$ or 5.4
5. $x = \sqrt{147} = 7\sqrt{3}$ or 12.1
6. $BC = \sqrt{90} = 3\sqrt{10}$ or 9.5
7. yes, $x = 8$
The sides are 9, 40, 41.

Lesson 60 - Similarity

AIM: What are the special relationships among the sides of a 30-60-90 triangle?

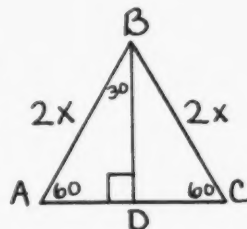
PERFORMANCE OBJECTIVES: The student will be able to...

1. Given the length of any side of a 30-60-90 triangle, compute the lengths of the other two sides.
2. Compute, to the nearest tenth, values of numbers such as $\sqrt{10}$, $7\sqrt{3}$.
3. Apply the rules that govern the lengths of the sides of the 30-60-90 triangle to numerical exercises.
4. Develop and apply formula for the height of an equilateral triangle: $h = \frac{s}{2}\sqrt{3}$.

VOCABULARY: 30-60-90 triangle, regular hexagon, center of a regular hexagon, apothem

MOTIVATION:

Consider equilateral $\triangle ABC$, each of whose sides is of length $2x$. The measure of each angle is 60. Draw altitude $\overline{BD}$. The measure of $\angle ABD$ is 30.



Can you express the lengths AD and BD in terms of x ?

DEVELOPMENT:

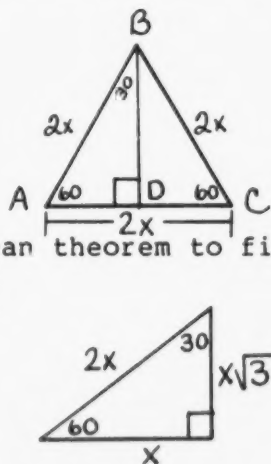
1. Given:

Find: AD, BD

AD = x . Use the Pythagorean theorem to find BD.

$$BD^2 + x^2 = 4x^2. \quad BD = x\sqrt{3}.$$

To summarize,



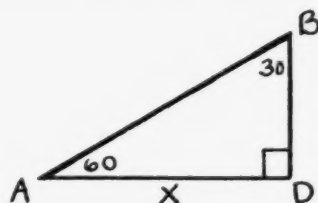
2. In a 30-60-90 triangle, the length of the side opposite the 30° angle is one-half the length of the hypotenuse. The length of the side opposite the 60° angle is equal to the length of the side opposite the 30° angle times $\sqrt{3}$.
3. Assume you are given the length of a leg of a 30-60-90 triangle.

a) Given: $AD = x$

Find: BD by multiplying

x by $\sqrt{3}$ ($x\sqrt{3}$)

Find: AB by doubling x ($2x$)



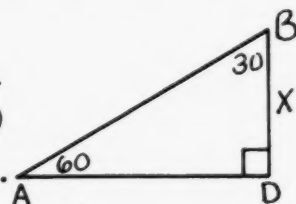
b) Given: $BD = x$

Find: AD by dividing

by $\sqrt{3}$

($\frac{x}{\sqrt{3}}$ should be simplified to $\frac{x\sqrt{3}}{3}$)

Find: AB by doubling AD ($\frac{2x\sqrt{3}}{3}$).



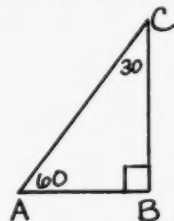
4. The following are numerical examples for $\triangle ABC$ (express answers first in simplest radical form and then to the nearest tenth).

Given: $m\angle CAB = 60$

$m\angle ACB = 30$

a) Given: $AC = 12$

Find: AB , BC ($AB = 6$, $BC = 6\sqrt{3} = 10.4$ to the nearest tenth.)



b) Given: $AB = 7$

Find: AC , BC ($AC = 14$, $BC = 7\sqrt{3}$ or 12.1)

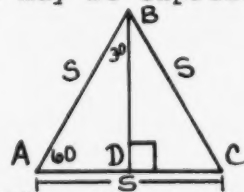
c) Given: $BC = \sqrt{30}$

Find: AB , AC ($AB = \sqrt{10}$ or 3.2, $AC = 2\sqrt{10}$ or 6.3)

5. The height of an equilateral triangle may be expressed in terms of the side length, s .

Given: equilateral $\triangle ABC$,
side length = s , height $\overline{BD}$

Find: BD



The length of the side opposite the 60° angle equals one half the length of the hypotenuse times $\sqrt{3}$. Therefore,
 $BD = \frac{s}{2}\sqrt{3}$.

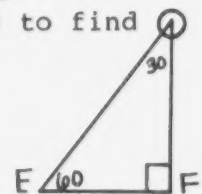
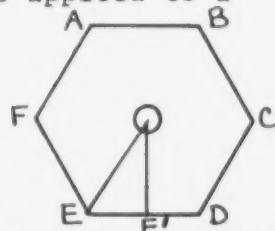
APPLICATIONS:

The rules for a 30-60-90 triangle can be applied to a hexagon.

Given: regular hexagon ABCDEF,
with side length 10, center O,
 $\overline{OF} \perp \overline{ED}$. Find OF' , OE .

$EF' = 5$, $OF' = 5\sqrt{3}$, $OE = 10$

Note that the length of the apothem, $\overline{OF}$, can be used to find the area of the regular hexagon.



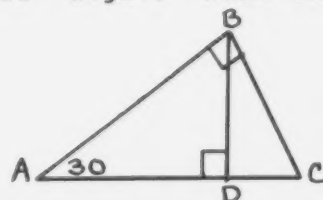
SUMMARY:

1. What is the relationship among the lengths of the sides of a 30-60-90 triangle?
2. What is the formula for expressing an altitude of an equilateral triangle in terms of the side length?

HOMEWORK:

Express all answers first in simplest radical form and then to the nearest tenth.

1. What is the length of an altitude of an equilateral triangle with side length 18?
2. A rhombus with side length 20 has a 60° angle. What is the length of each diagonal?
3. Given: $m\angle A = 30$, right $\triangle ABC$
 $\overline{BD} \perp \overline{AC}$, $DC = 4$
Find: AD , AB , BC , BD , AC



4. Find the apothem of a regular hexagon of side length 12.
5. The side opposite the 60° angle of a 30-60-90 triangle has length 32. What are the lengths of the other two sides?

HOMEWORK SOLUTIONS:

1. $9\sqrt{3}$ or 15.6

2. The shorter diagonal has length 20. The longer diagonal has length $20\sqrt{3}$ or 34.6.
3. $AD = 12$, $AB = 8\sqrt{3}$ or 13.9, $BC = 8$, $BD = 4\sqrt{3}$ or 6.9, $AC = 16$
4. $6\sqrt{3}$ or 10.4
5. leg = $\frac{32\sqrt{3}}{3}$ or 18.5 hypotenuse = $\frac{64\sqrt{3}}{3}$ or 37.0

Lesson 61 - Similarity

AIM: What are the special relationships among the sides of the 45-45-90 triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Compute the lengths of the other two sides of a 45-45-90 triangle, given the length of any side.
2. Compute $\sqrt{2}$ to the nearest thousandth.
3. Apply the rules that govern the lengths of the sides of the isosceles right triangle in numerical exercises.
4. Develop and apply the formula for the diagonal of a square: $d = s\sqrt{2}$.

VOCABULARY: 45-45-90 triangle, isosceles right triangle

MOTIVATION:

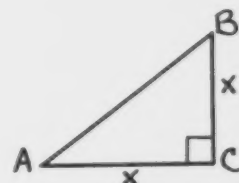
What is the distance between the opposite vertices of a square plot of land which is 6 miles on a side?

DEVELOPMENT:

1. Given: isosceles right $\triangle ACB$,

$$AC = BC = x$$

Find: AB



Use the Pythagorean theorem, $AB^2 = x^2 + x^2$, $AB = x\sqrt{2}$.

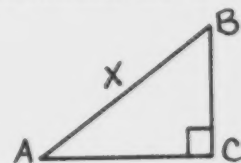
Find $\sqrt{2}$ to the nearest thousandth (1.414).

2. In a 45-45-90 triangle, the length of the hypotenuse equals the length of a leg times $\sqrt{2}$.
3. Assume you are given the length of the hypotenuse of a 45-45-90 triangle.

Given: isosceles right $\triangle ACB$, $AB = x$

Find: AC (or BC) by dividing x by $\sqrt{2}$.

$$AC = \frac{x}{\sqrt{2}}, \text{ which can be simplified to } \frac{x\sqrt{2}}{2}.$$



4. The following are numerical exercises (express answers first in simplest radical form and then to the nearest hundredth).

Given: isosceles right $\triangle ABC$, with hypotenuse $\overline{AC}$.

a) Given: $AB = 12$

Find: AC ($12\sqrt{2}$ or 16.97)

b) Given: $AC = 20$

Find: AB ($AB = 10\sqrt{2}$ or 14.14)

5. The length of the diagonal of a square may be expressed in terms of the length of a side, s .

Given: square $ABCD$, $AD = s$

Find: BD

The length of the diagonal (hypotenuse) equals the length of a side times $\sqrt{2}$, that is, $s\sqrt{2}$. For example, find the length of the diagonal of a square of side length 30.

Length = $30\sqrt{2}$ or 42.43.

APPLICATIONS:

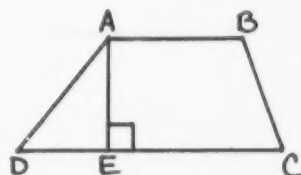
The rules for a 45-45-90 triangle can be applied to isosceles trapezoids.

Given: isosceles trapezoid $ABCD$,

$\overline{AB} \parallel \overline{DC}$, $AD = BC$, $m\angle D = 45$,

$AB = 16$, $DC = 24$

Find: AD ($DE = 4$, $AD = 4\sqrt{2}$ or 5.66)



SUMMARY:

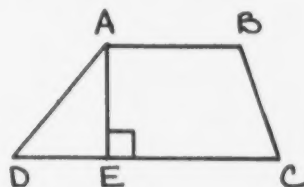
1. What is the relationship among the lengths of the sides of a 45-45-90 triangle?
2. What is the formula for expressing the length of the diagonal of a square in terms of the side length?

HOMEWORK:

Express answers first in simplest radical form and then to the nearest hundredth.

1. What is the length of a side of a square whose diagonal measures $5\sqrt{2}$?

2. What is the length of the side of a square whose diagonal measures 60?
3. What is the length of a diagonal of a square whose side measures 40?
4. Find the area of a square with diagonal of length 50.
5. Given: isosceles trapezoid ABCD
 $\overline{AB} \parallel \overline{DC}$, $AD = BC$, $m\angle D = 45^\circ$,
 $AB = 30$, $DC = 40$
 Find: AD



HOMEWORK SOLUTIONS:

1. 5
2. $30\sqrt{2}$ or 42.43
3. $40\sqrt{2}$ or 56.57
4. side = $25\sqrt{2}$
 $\text{area} = (\text{side})^2 = (25\sqrt{2})^2 = 1250 \text{ sq. units}$
5. $DE = 5$
 $AD = 5\sqrt{2}$ or 7.07